14.X.22

Compactness (topologizing the Extreme Value Theorem)

A space is **compact** if every open cover has a finite subcover.

E.g.
- Every finite space
- Trivial topologies
- Discrete spaces iff finite
- Closed & bounded subspaces of $\mathbb{R}^n$ (more later...)

Warmup Prop Suppose $x_i \to x \in X$, an arbitrary top space. Then $A = \{x_i \mid i \in \mathbb{N}\} \cup \{x\}$ is compact.

Pf Suppose $\mathcal{U}$ is an open cover of A. Then $\exists U \in \mathcal{U}$ s.t. $x \in U$. Since $x_i \to x$, all but finitely many $x_i \in U$, say ~~$x_0, x_1, \ldots, x_N \notin U$~~ $x_{N+1}, x_{N+2}, \ldots \in U$. Choose $U_0, \ldots, U_N \in \mathcal{U}$ containing $x_0, \ldots, x_N$. Then $\{U_i \mid i = 0, \ldots, N\} \cup \{U\}$ is a finite subcover of $\mathcal{U}$. □

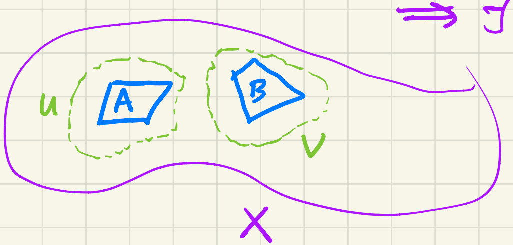
THM If $f : X \to Y$ is cts and X is compact, then fX is compact.

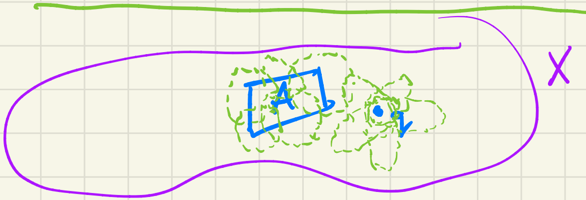
Pf Suppose $\mathcal{U}$ is a cover of fX by open subsets of Y. Then $f^{-1}\mathcal{U} := \{f^{-1}U \mid U \in \mathcal{U}\}$ is an open cover of X. By compactness of X, $f^{-1}\mathcal{U}$ has an open subcover $\{f^{-1}U_0, \dots, f^{-1}U_N\}$. Thus $\{U_0, \dots, U_N\}$ form a finite subcover of $\mathcal{U}$. $\square$

Cor If X is compact and $f : X \xrightarrow{\cong} Y$, then Y is compact. $\square$

Two lemmata:

Separation Lemma X Hausdorff, $A, B \subseteq X$ disjoint compact subspaces $\Rightarrow \exists\, U, V \subseteq X$ open, disjoint s.t. $A \subseteq U$, $B \subseteq V$.





Pf First suppose $B = \{q\}$. For each $p \in A$ take $U_p \ni p$, $V_p \ni q$ disjoint nbhds. Since A is compact, $\exists\, p_1, \dots, p_k \in X$ s.t. $\{U_{p_1}, \dots, U_{p_k}\}$ cover A.

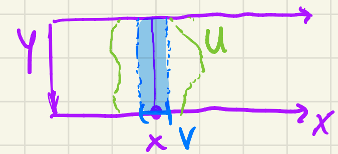
Set $\mathbb{U} = U_{p_1} \cup \dots \cup U_{p_k}$, $\mathbb{V} = V_{p_1} \cap \dots \cap V_{p_k}$

$\mathbb{U} \supseteq A$, $\mathbb{V} \ni q$

open, disjoint ✓.

Now take arbitrary $A, B \subseteq X$ compact, disjoint. By the above argument, $\forall q \in B$ $\exists\, U_q, V_q \subseteq X$ disjoint open s.t. $A \subseteq U_q$, $q \in V_q$. Since B compact, $\exists\, q_1, \dots, q_m \in B$ s.t. $\{V_{q_1}, \dots, V_{q_m}\}$ cover B. Set $U = U_{q_1} \cap \dots \cap U_{q_m}$, $V = V_{q_1} \cup \dots \cup V_{q_m}$. These work. □

Tube Lemma X a space, Y compact. If $x \in X$ and $U \subseteq X \times Y$ open containing $\{x\} \times Y$, then $\exists$ nbhd V of x s.t. $V \times Y \subseteq U$.



Pf For each $y \in Y$ $\exists\, V_y \times W_y \subseteq X \times Y$ nbhd of (x, y) inside U. By compactness of Y, $\exists\, y_1, \dots, y_k \in Y$ s.t. $(V_{y_1} \times W_{y_1}) \cup \dots \cup (V_{y_k} \times W_{y_k})$ cover $\{x\} \times Y$.

Set $V = V_{y_1} \cap \dots \cap V_{y_k}$; take $V \times Y$. □

Properties of compact spaces

(a) closed $\subseteq$ compact is compact
(b) compact $\subseteq$ H'ff is closed
(c) compact $\subseteq$ metric is bounded
(d) finite products of compact spaces are compact
(e) quotients of compact spaces are compact.

Pf Read 4.36. $\square$

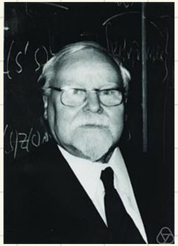
Rmk (d) is true for infinite products as well: Tychonoff's Theorem (1930/1935).

Compact subspaces of $\mathbb{R}$:

Thm Every closed bounded interval in $\mathbb{R}$ is compact.

Pf Let $[a,b] \subseteq \mathbb{R}$ be such an interval and let $\mathcal{U}$ cover $[a,b]$ by opens in $\mathbb{R}$.

Set
$$X := \left\{ x \in (a,b] \;\middle|\; \begin{array}{l}[a,x] \text{ is covered by} \\ \text{finitely many sets of } \mathcal{U}\end{array} \right\}$$



Andrey Tikhonov
1906 – 1993
topology, functional analysis, math-phys, geophysics

1948: magnetotellurics

and $c = \sup X$. WTS $c = b$.

\ check X nonempty, bdd above so c exists.

Take $c \in U_0 \in \mathcal{U}$. Since U_0 open, $\exists \varepsilon > 0$ s.t. $(c-\varepsilon, c] \subseteq U_0$.

Since $c = \sup X$, $\exists x \in X$ s.t. $c - \varepsilon < x < c$. Thus $\exists U_1, \dots, U_k \in \mathcal{U}$ covering $[a, x]$, whence $[a, c] \subseteq U_1 \cup \dots \cup U_k \cup U_0$. Suppose for contradiction $c < b$. Because U_0 open $\exists x > c$ s.t. $[a, x] \subseteq U_1 \cup \dots \cup U_k \cup U_0$, contradicting $c = \sup X$. $\square$

Heine-Borel Thm $K \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

Pf ($\Rightarrow$) (b) + (c) of properties (so true for any metric space).

($\Leftarrow$) Suppose $K \subseteq \mathbb{R}^n$ closed, bdd: $K \subseteq [-R, R]^n$ for some $R > 0$.

Have $[-R, R]$ compact by previous thm, so $[-R, R]^n$ compact by (d).

Now K is a closed subspace of a compact space, hence compact by (a). $\square$

Thm (topological EVT) X compact, $f: X \to \mathbb{R}$ cts implies f is bounded and attains its max & min values.

Pf By THM, $fX \subseteq \mathbb{R}$ is compact, hence closed and bounded so contains its inf & sup. $\square$

TPS What is the image of a compact connected space X under a cts map $f: X \to \mathbb{R}$?   **A** closed bdd interval, $\{x\}$, $\emptyset$

Sequential & limit point compactness (précis)

Call a space X **limit point compact** when every infinite subspace of X has a limit point in X

sequentially compact every sequence in X has a convergent subsequence

Facts

- Compact $\Rightarrow$ limit point compact
- first countable H'ff + limit point compact $\Rightarrow$ sequentially compact
- (metric or second countable) + seq compact $\Rightarrow$ compact

?? — H'ff

So for metric or second countable spaces (e.g. subspaces of manifolds)
all three notions of compactness are equivalent!

Corollaries:

Bolzano-Weierstrass Thm Bounded sequences in $\mathbb{R}^n$ have convergent subsequence.

Thm $X \subseteq \mathbb{R}^n$ w/ Euclidean metric is complete iff it is closed in $\mathbb{R}^n$.

Thm Every compact metric space is complete.

Aside At the cost of extra abstraction, (ultra)filters (replacing sequences) make this story (and the "H'ff $\Rightarrow$ sequences have ! limits" story) nicer.
See Ch. 3 of Topology: A Categorical Approach.

Closed map lemma (easy! simple! powerful!) For F: compact $\rightarrow$ H'ff cts
(a) F is closed
(b) F surj $\Rightarrow$ F quotient
(c) F inj $\Rightarrow$ F top embedding
(d) F bij $\Rightarrow$ F homeomorphism

Pf Fix $F: X \to Y$ cts. If $A \subseteq X$ closed, then A is compact, so
(X compact, Y H'ff)

FA is compact, so FA is closed. We've already noted the other properties for closed maps. $\square$

E.g. (1) $\omega: [0,1]^n \longrightarrow T^n$ is a quotient map.
$x \longmapsto \exp(2\pi i x)$

(2) The composite $S^n \xhookrightarrow{\iota} \mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{q} \mathbb{RP}^n$ is a quotient map $q\iota : S^n \longrightarrow \mathbb{RP}^n$ by the closed map lemma. We have $q\iota(x) = q\iota(y) \iff x = \pm y$, so $\mathbb{RP}^n \cong S^n/x \sim -x$. Since quotients of compact spaces are compact, $\mathbb{RP}^n$ is compact ☺

(3) By CML, a cts surj map $q: \bar{\mathbb{B}}^n \longrightarrow S^n$ injective on $\mathbb{B}^n$ and taking S^{n-1} to a pt $\Longrightarrow \bar{\mathbb{B}}^n / S^{n-1} \cong S^n$.