- $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ — $GL_n(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ subspace top
- subgroups of topological gps
- any group with the discrete topology

⚠ $(\mathbb{R}, +)$ and $(\mathbb{R}^{disc}, +)$ are isomorphic as groups but not as topological groups. (Condensed mathematics?)

12.X.22

Defn A space X is (topologically) homogeneous when $\forall x, y \in X$ $\exists$ homeo $\varphi : X \to X$ with $\varphi(x) = y$.

{ X looks the same from every point

Prop Topological groups are homogeneous.

Pf For $g \in G$, define $L_g : G \to G$, $h \mapsto gh$. Since m is cts, so is L_g; it has cts inverse $L_{g^{-1}}$ so each L_g is a homeo. For $g, g' \in G$, $L_{g'g^{-1}}$ is a homeo w/ $L_{g'g^{-1}}(g) = (g'g^{-1})g = g'$. $\square$

Group actions

For G a topological gp, X a space, we are interested in continuous group actions $G \times X \longrightarrow X$

$$(g, x) \longmapsto g \cdot x$$

s.t. $e \cdot x = x \ \forall x \in X$, $g \cdot (h \cdot x) = (gh) \cdot x \ \forall g, h \in G, x \in X$.

(This is a left action; can also talk about right actions.)

Prop Continuous actions act by homeomorphisms:

$$\forall g \in G, \quad g \cdot : X \xrightarrow{\cong} X$$
$$x \longmapsto g \cdot x$$

Pf $g^{-1} \cdot$ is a cts two-sided inverse. $\square$

E.g.
- $GL_n(\mathbb{R}) \circlearrowright \mathbb{R}^n$ transitively on nonzero vectors
- $\mathbb{R}^\times \circlearrowright \mathbb{R}^n \setminus \{0\}$ with $(\mathbb{R}^n \setminus \{0\}) / \mathbb{R}^\times \cong \mathbb{RP}^{n-1}$ orbit space with quotient topology

  $\lambda \cdot x = (\lambda x_1, \dots, \lambda x_n)$

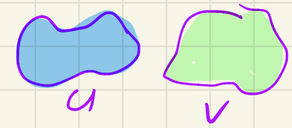
$\mathbb{R} \setminus \{0\}$

- $\mathbb{R}^n/\mathbb{Z}^n \cong T^n$

$$(x_1, \dots, x_n) + \mathbb{Z}^n \mapsto (\exp(2\pi i x_1), \dots, \exp(2\pi i x_n))$$

Connectedness (topologizing the Intermediate Value Thm)

A space X is disconnected when $X = U \cup V$ for $U, V \subseteq X$ open, nonempty, disjoint; otherwise X is connected.



Prop X is connected iff $X, \varnothing$ are the only clopen subsets of X.

clopen = closed and open

Pf ($\Rightarrow$) Suppose X conn'd, $U \subseteq X$ clopen. Then $V = X \setminus U$ is also clopen and $X = U \cup V$. Thus one of $U, V = X$, the other is $\varnothing$.

($\Leftarrow$) Suppose X is disconnected and $X = U \cup V$ witnesses this. Then both U, V are also closed (b/c complements V, U are open) and neither is empty so neither is X. $\square$

Prop Suppose X is connected. Then every cts map to a discrete space is constant.

Pf Nothing to prove if $X = \emptyset$. Suppose $X \neq \emptyset$, Y discrete, $f : X \to Y$ cts. Choose $x \in X$ and let $c = f(x)$. Since $\{c\} \subseteq Y$ is clopen, $f^{-1}\{c\} \subseteq X$ is clopen $\Longrightarrow f^{-1}\{c\} = X$. $\square$

E.g.

- $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is disconnected 
- $\mathbb{Q}^2 \subseteq \mathbb{R}^2$ is disconnected:
$$\mathbb{Q}^2 = \{(x,y) \in \mathbb{Q}^2 \mid x < \sqrt{2}\} \cup \{(x,y) \in \mathbb{Q}^2 \mid x > \sqrt{2}\}.$$
- Directly proving connectedness of, say, $\mathbb{R}$ or S^1 or ... — harder.

THM If $f : X \to Y$ is cts and X is conn'd, then fX is conn'd.

Pf WLOG, assume f surjective. For proof by contrapositive, suppose Y disconn'd, witnessed by $Y = U \cup V$. Then $f^{-1}U$, $f^{-1}V$ disconnect X. $\square$

Cor Connectedness is a topological property (preserved by homeo).

See 4.9 for properties of connected spaces.

Connected subsets of $\mathbb{R}$:

$J \subseteq \mathbb{R}$ is an interval if $|J| > 1$ and $\forall a, b \in J$ if $a < c < b$ for some $c \in \mathbb{R}$, then $c \in J$.

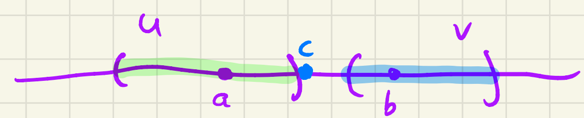
Fact J is of the form $[a,b]$, (a,b), $[a,b)$, $(a,b]$, $(-\infty, b]$, $(-\infty, b)$, $[a,\infty)$, (a,∞), or $(-\infty, \infty)$.

Prop A nonempty subset of $\mathbb{R}$ is conn'd iff it's a singleton or an interval.

Pf Singletons ✓ so assume J has at least two pts.

($\Leftarrow$) If J is not conn'd, then $\exists\, U, V \subseteq \mathbb{R}$ open with $U \cap J$, $V \cap J$ disconnecting J. WLOG, $a \in U \cap J$, $b \in V \cap J$, $a < b$. Since J is an interval, $[a,b] \subseteq J$.

Pick $\varepsilon > 0$ s.t. $[a, a+\varepsilon) \subseteq U \cap J$



and $(b-\varepsilon, b] \subseteq V \cap J$.

Set $c = \sup(U \cap [a,b])$. Then $a+\varepsilon \leq c \leq b-\varepsilon \Rightarrow a<c<b$

$\Rightarrow c \in J \subseteq U \cup V$. If $c \in U$, then $\exists \delta > 0$ s.t. $(c-\delta, c+\delta) \subseteq U$ ☠

If $c \in V$, then $\exists \delta > 0$ s.t. $(c-\delta, c+\delta) \subseteq V$, disjoint from U ☠

Thus J is conn'd.

($\Rightarrow$) If J is not an interval, then $\exists\, a, b \in J$ and $a<c<b$ with $c \notin J$.

The sets $(-\infty, c) \cap J$ and $(c, \infty) \cap J$ disconnect J. $\square$

Thm (IVT) If X is conn'd, $f: X \to \mathbb{R}$ cts, $p, q \in X$, then f attains every value b/w $f(p)$ and $f(q)$.

Pf fX is connected and hence an interval. $\square$

Application (dimension $n=1$ of the Brouwer fixed point theorem)

Every cts function $f: [-1,1] \to [-1,1]$ has a fixed point (x s.t. $f(x) = x$).

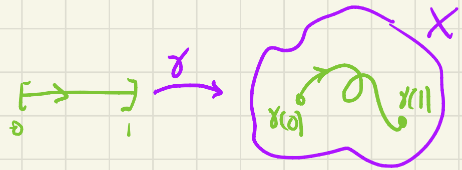
Pf Assume for ☠ f has no fixed pt. Then we can form a cts fn

$$g : [-1,1] \longrightarrow \{\pm 1\}$$
$$x \longmapsto \frac{x - f(x)}{|x - f(x)|}.$$

Since $f(-1) > -1$, $g(-1) = -1$.
Since $f(1) < 1$, $g(1) = 1$.
☠ since $[-1,1]$ is conn'd and thus g must be constant! $\square$

Path connectedness

A **path** in X is a cts function $\gamma : [0,1] \longrightarrow X$.



We say γ is a path in X from $\gamma(0)$ to $\gamma(1)$.

X is **path connected** when $\forall p, q \in X$ $\exists$ path γ in X from p to q.

See 4.13 for basic properties.

Thm If X is path connected, then X is connected.

Pf Suppose X path conn'd and $f: X \to \{0,1\}$ is cts. (WTS f is constant.)

Fix $x_0 \in X$ and for each $x \in X$ choose a path $\gamma: [0,1] \to X$ from x to x_0.

We have

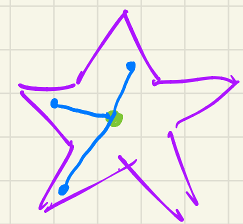
$$\begin{tikzcd} {[0,1]} \arrow[r, "\gamma"] \arrow[dr, "f\gamma"'] & X \arrow[d, "f"] \\ & \{0,1\} \end{tikzcd}$$

and since $[0,1]$ is conn'd, $f\gamma$ is constant.

Thus $f(x) = f\gamma(0) = f\gamma(1) = f(x_0)$ so f is constant. $\square$

E.g. Path conn'd (hence conn'd) spaces:

- $\mathbb{R}^n$
- Convex and star convex subsets of $\mathbb{R}^n$
- $\mathbb{R}^n \setminus \{0\}$ for $n \geq 2$
- S^n for $n \geq 1$
- T^n



E.g. Set $T_0 = \{0\} \times [-1,1] \subseteq \mathbb{R}^2$

$T_+ = \{(x, \sin(1/x)) \mid x \in (0, 2/\pi)\} \subseteq \mathbb{R}^2$

The topologist's sine curve is $T = T_0 \cup T_+$

T is conn'd but not path conn'd.

Components & path components

A component of X is a maximal nonempty conn'd subset of X
wrt $\subseteq$

Prop The components of X form a partition of X. $\square$

See 4.20 for properties.

A path component of X is a maximal nonempty path conn'd subset of X.

See 4.21 for properties.

Write $\pi_0 X$ for the set of path components of X

Q When are components & path components the same?

A When X is locally path connected.

admits a basis of path-conn'd open subsets

We can also talk about locally connected spaces i.e. those admitting a basis of conn'd open subsets.

Facts
- Every mfld (w/ or w/o ∂) is locally path conn'd.
- Locally path conn'd $\Rightarrow$ locally conn'd.
- Locally path conn'd $\Rightarrow$ path components = components
  (so conn'd iff path conn'd)