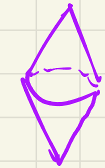
10.X.22

(ii) The (unreduced) suspension of X is

$$\tilde{\Sigma} X := X \times [0,1] \Big/ \begin{matrix} (x,0) \sim (x',0) \\ (x,1) \sim (x',1) \end{matrix}$$



$\tilde{\Sigma} X$

(e) Set $\mathbb{RP}^n := \{$1-dim'l linear subspaces of $\mathbb{R}^{n+1}\}$ and define

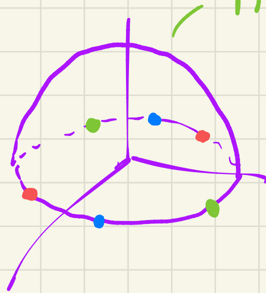
$$q : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$$
$$x \longmapsto \operatorname{span}\{x\} = \{\lambda x \mid \lambda \in \mathbb{R}\}.$$

Give $\mathbb{RP}^n$ the quotient topology wrt q.

$q : X \twoheadrightarrow S$ endow S w/ quotient top rel to q: $U \subseteq S$ iff $q^{-1}U \subseteq X$ open

<u>$n=2$</u>

$\mathbb{RP}^2 \cong$



$\Big/ (x,y,0) \sim (-x,-y,0) \cong \bar{\mathbb{B}}^2 \Big/$ for $x \in \partial \bar{\mathbb{B}}^2 = S^1$, $x \sim -x$

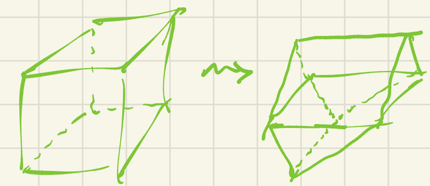
(f) For $X, Y \neq \emptyset$, the **join** of X, Y is

$$X * Y := X \times Y \times [0,1] \Big/ \begin{array}{l}(x, y_1, 0) \sim (x, y_2, 0)\\ (x_1, y, 1) \sim (x_2, y, 1)\end{array}$$

TPS (i) Draw $[0,1] * [0,1]$

(ii) What is $X * \{pt\}$? CX

(iii) What is $X * S^0$? $\tilde{\Sigma} X$



$X \times \{pt\} \times [0,1)$

To do: • For X Hausdorff / a mfld, when is $X/\sim$ H'ff / a mfld?

- Universal property of quotients.
- Recognizing quotient maps.
- Gluing
- Topological groups, group actions

Prop Locally Euclidean quotients of second countable spaces are second countable.

Pf Consider $q : P \twoheadrightarrow M$ a quotient map. Cover M by

2nd countable (P) loc Euclidean (M)

coordinate balls to get $\mathcal{U}$. Then $\{q^{-1}U \mid U \in \mathcal{U}\}$ is an open cover of P $\Rightarrow$ it has a countable subcover. Let $\mathcal{U}' \subseteq \mathcal{U}$ be countable w/ $\{q^{-1}U \mid U \in \mathcal{U}'\}$ covering P. Then $\mathcal{U}'$ is a countable cover of M by coordinate balls. Each ball is 2nd countable, so M is second countable. $\square$

Prop If $X \to X/\sim$ is an open map, then $X/\sim$ is H'ff iff $\underbrace{\sim \subseteq X \times X}$ is closed.

$\sim = \{(x,y) \mid x \sim y\}$

Pf Read 3.57, 3.58. $\square$

Prop For $f: X \to Y$ cts and open or closed

(a) f inj $\Rightarrow$ embedding

(b) f surj $\Rightarrow$ quotient

(c) f bij $\Rightarrow$ homeo. (Read pp. 69–71.) $\square$

③ **Thm** Suppose $q: X \to Y$ is a quotient map. Then for any space Z and fn $f: Y \to Z$, f is cts iff $f \circ q$ is cts:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "fq"'] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

cts $\Leftrightarrow$

The quotient top on Y is the only topology satisfying this condition.

Pf ($\Rightarrow$) f, q cts always implies fq cts.

($\Leftarrow$) If fq cts, then $\forall U \subseteq Z$ open, $(fq)^{-1}U = q^{-1}(f^{-1}U)$ open.
By defn of quotient top, this implies $f^{-1}U \subseteq Y$ open, so f cts.

(uniqueness) Dualize the uniqueness proof for the subspace top. $\square$

Cor

$$\begin{tikzcd} X \arrow[r, "q"', "\text{quotient}"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\tilde{f}"] \\ & Z \end{tikzcd}$$

A cts map $\tilde{f}$ making the diagram commute exists iff f is cts and constant on the $\underbrace{\text{fibers of } q}_{q^{-1}\{y\}}$.

i.e. $q(x) = q(x') \Rightarrow f(x) = f(x')$. $\square$

E.g. A cts function on $\mathbb{R}$ descends to $\mathbb{R}/\mathbb{Z} \cong S^1$ iff it is 1-periodic.

Thm The corollary is a universal property for quotient spaces specifying Y up to homeomorphism.

TPS What are the hypotheses on the above thm for $X \xrightarrow{q'} Y'$ s.t. $\exists!$ homeo

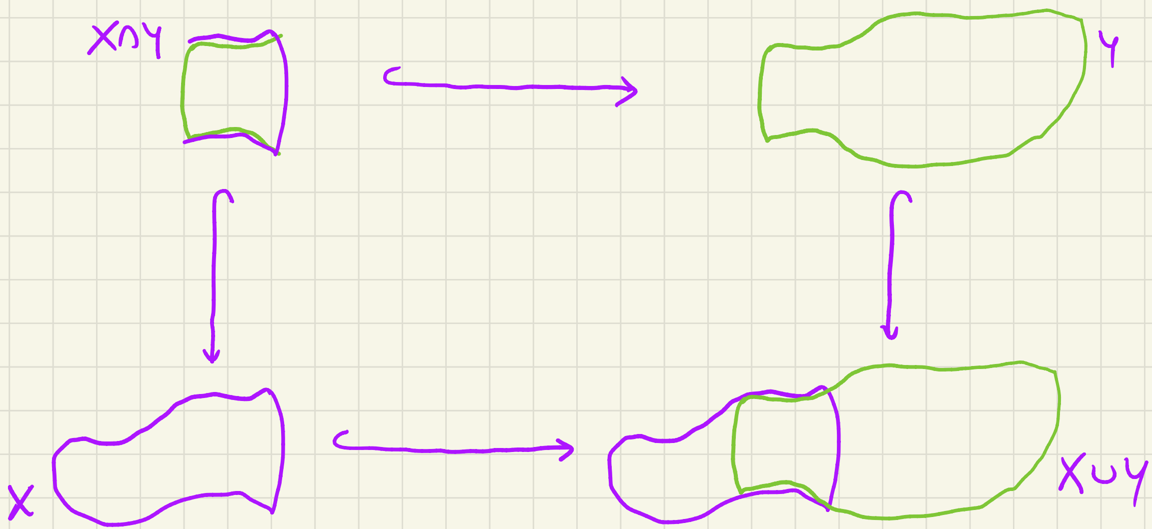
$$\begin{tikzcd} & X \arrow[dl, "q"'] \arrow[dr, "q'"] & \\ Y \arrow[rr, "\cong"'] & & Y' \end{tikzcd}$$

?

A q, q' have the same fibers

Pushouts / Adjunction spaces / Gluing

Q How do we reconstruct $X \cup Y$ from X, Y, and $X \cap Y$?



Generalize to

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & X \cup_A Y \end{tikzcd}$$

($\ulcorner$ = "pushout")

$X \cup_A Y := X \sqcup Y \,/\, f(a) \sim g(a)$

or $X \underset{f\ \ g}{\cup} Y$

inclusion into $X \sqcup Y$ followed by quotient

Thm

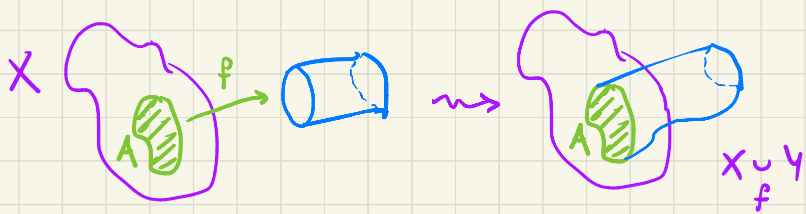
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & X \cup_A Y \arrow[dr, dashed, "\exists!"] & \\ & & Z \end{tikzcd}$$

i.e. if the outer square commutes, then $\exists!\ X \cup_A Y \to Z$ making the triangles commute.

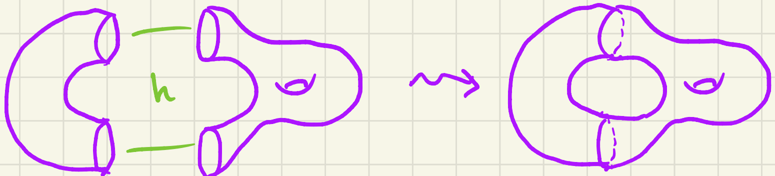
This specifies $X \cup_A Y$ up to (appropriately unique) homeomorphism. $\square$

Special case $A \subseteq X$ and $f : A \to Y$ cts.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & X \cup_f Y \end{tikzcd}$$



Extra special case: M, N n-dimensional mflds w/ boundary, $h : \partial M \xrightarrow{\cong} \partial N$. Then $M \cup_h N$ is an n-manifold w/o boundary.



M $\quad$ N $\quad$ $M \cup_h N$

(Read 3.79 for details.)

Later: Attaching cells.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d] \\ \bar{\mathbb{B}}^n \arrow[r] & \underbrace{X \cup_\varphi \bar{\mathbb{B}}_n}_{} \end{tikzcd}$$

X w/ an n-cell attached by φ

Topological groups

Groups have multiplication $m : G \times G \to G$ and inversion $i : G \to G$ functions.

If G is a space and m, i continuous, then call G a topological group.

Eg.
- $(\mathbb{R}, +)$, $(\mathbb{C}, +)$
- $(\mathbb{R}^\times = \mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{C}^\times, \cdot)$

- $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ $\quad GL_n(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ subspace top
- subgroups of topological gps
- any group with the discrete topology

(!) $(\mathbb{R}, +)$ and $(\mathbb{R}^{disc}, +)$ are isomorphic as groups but not as topological groups. (Condensed mathematics?)

Defn A space X is (topologically) homogeneous when $\forall x, y \in X$ $\exists$ homeo $\varphi: X \to Y$ with $\varphi(x) = y$.

{ X looks the same from every point

Prop Topological groups are homogeneous.

Pf For $g \in G$, define $L_g: G \to G$, $h \mapsto gh$. Since m is cts, so is L_g; it has cts inverse $L_{g^{-1}}$ so each L_g is a homeo. For $g, g' \in G$, $L_{g'g^{-1}}$ is a homeo w/ $L_{g'g^{-1}}(g) = (g'g^{-1})g = g'$. $\square$