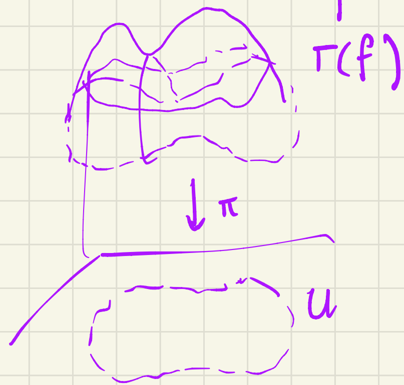
$$\Gamma(f) := \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^k$$

is a manifold homeomorphic to U (via projection)



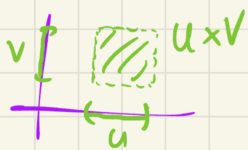
__Note__ Injectivity of π = "vertical line test"

__Products__

7.X.22

For spaces X, Y, what topology should $X \times Y$ have?

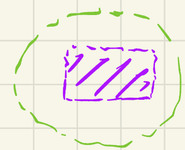
① The __product topology__ on $X \times Y$ is the topology generated by the basis $\mathcal{B} = \{U \times V \mid U \subseteq X \text{ \& } V \subseteq Y \text{ open}\}$



__Moral Exercise__ Check that $\mathcal{B}$ is a basis!

So open sets in $X \times Y$ are unions of "product open sets".

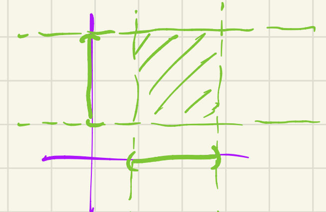
E.g. The product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ equals the Euclidean topology on $\mathbb{R}^2$:

 & 

② The product topology is the coarsest topology on $X \times Y$ s.t. both projection maps

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

are cts.



③ Thm Given cts maps $X \xleftarrow{f} Z \xrightarrow{g} Y$ $\exists!$ cts map $(f,g): Z \to X \times Y$ s.t. $\pi_X \circ (f,g) = f$, $\pi_Y \circ (f,g) = g$.

exists unique

I.e.

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\exists!\,(f,g)"] \arrow[dr, bend left, "g"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}.$$

Moreover, if $X \xleftarrow{p_X} W \xrightarrow{p_Y} Y$ satisfies

$$\begin{tikzcd} & Z \arrow[dl, bend right] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left] & \\ X & W \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

then there is a unique homeomorphism s.t.

$$\begin{tikzcd} X \times Y \arrow[r, "\cong"] \arrow[d, "\pi_X"'] \arrow[dr, "\pi_Y"' near start] & W \arrow[d, "p_Y"] \arrow[dl, "p_X" near start] \\ X & Y \end{tikzcd}$$

commutes.

Pf The function $(f, g) : Z \longrightarrow X \times Y$

$z \longmapsto (f(z), g(z))$

makes the diagram commute. If

$\pi_X(x,y) = f(z)$ & $\pi_Y(x,y) = g(z)$, then

$x = f(z)$ & $y = g(z)$,

so this is the only such function. For continuity, it suffices to check $(f,g)^{-1}(U \times V)$ open for $U \subseteq X$, $V \subseteq Y$ open.

$= f^{-1}U \cap g^{-1}V$ ✓

Uniqueness: moral exc/reading. □

Slogan (Continuous) functions into a product are determined by their (continuous) coordinate functions.

Here for $f : Z \longrightarrow X_1 \times \cdots \times X_n$, $f_i := \pi_i \circ f$ is the i-th coord fn.

$z \longmapsto (f_1(z), \ldots, f_n(z))$

(See pp. 61–63 for more "productive" facts.)

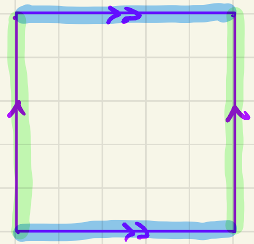
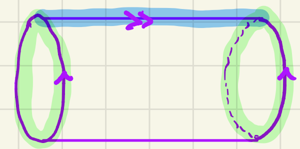
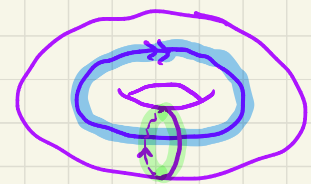
TPS Is the product of manifolds M, N a manifold? Of what dimn?

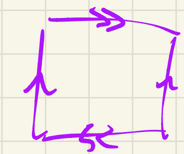
$(p, q) \in M \times N$ $\quad p \in U \cong \mathbb{R}^d$ $\quad q \in V \cong \mathbb{R}^e$ $\quad \underset{(p,q)}{U \times V} \subseteq M \times N$

$\cong \mathbb{R}^d \times \mathbb{R}^e = \mathbb{R}^{d+e}$

E.g. the n-torus is $T^n := \underbrace{S^1 \times \cdots \times S^1}_{n \text{ factors}} = (S^1)^n$.

Note that $T^2 \subseteq \mathbb{R}^4$, but we can still visualize it:

$T^2 \cong$  $\cong$  $\cong$ 

 $= K$

See 3.22 & 3.36 for an explicit homeomorphism.

Infinite products

The product topology on $\prod_{\alpha\in A} X_\alpha$ has basis

$$\mathcal{B} := \left\{ \prod_{\alpha\in A} U_\alpha \;\middle|\; \begin{array}{l} \forall \alpha\in A,\ U_\alpha\subseteq X_\alpha \text{ is open}, \\ \underbrace{\text{almost every}}_{\text{all but finitely many}}\ U_\alpha = X_\alpha \end{array} \right\}$$

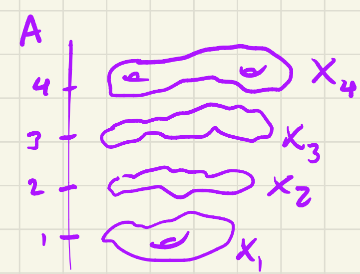
This satisfies the correct universal property

$$\begin{tikzcd} & Z \arrow[d, dashed, "\exists!\,(f_\alpha)_{\alpha\in A}"] \arrow[ddl, bend right, "f_\alpha\ \forall\alpha"'] \\ & \prod_{\alpha\in A} X_\alpha \arrow[dl, "\pi_\alpha"] \\ X_\alpha & \end{tikzcd}$$

Coproducts (disjoint unions)

Given an indexed family $\{X_\alpha \mid \alpha\in A\}$ of sets/spaces,

$$\coprod_{\alpha\in A} X_\alpha := \{(x,\alpha) \mid \alpha\in A,\ x\in X_\alpha\}.$$



For $\beta\in A$, have

$$\iota_\beta : X_\beta \longrightarrow \coprod_{\alpha\in A} X_\alpha.$$
$$x \longmapsto (x,\beta)$$

① The coproduct topology on $\coprod_{\alpha\in A} X_\alpha$ has open sets $U \subseteq \coprod_{\alpha\in A} X_\alpha$ s.t. $U \cap X_\alpha \subseteq X_\alpha$ open $\forall \alpha \in A$. I.e. $U = \coprod_{\alpha\in A} U_\alpha$ for $U_\alpha \subseteq X_\alpha$ open.

② This is the finest topology on $\coprod X_\alpha$ s.t. ι_α is cts $\forall \alpha \in A$.

③ Thm

$$\begin{tikzcd} & Y \\ & \coprod_{\alpha\in A} X_\alpha \arrow[u, dashed, "\exists! f"'] \\ X_\alpha \arrow[uur, bend left, "\forall \alpha\ f_\alpha"] \arrow[ur, "\iota_\alpha"'] & \end{tikzcd}$$

i.e. if $\forall \alpha \in A$, $f_\alpha : X_\alpha \to Y$ is cts, then $\exists!$ cts $f : \coprod_{\alpha\in A} X_\alpha \to Y$ s.t.

$$f\iota_\alpha = f_\alpha \quad \forall \alpha \in A.$$

This property characterizes $\coprod_{\alpha\in A} X_\alpha$.

Quotients

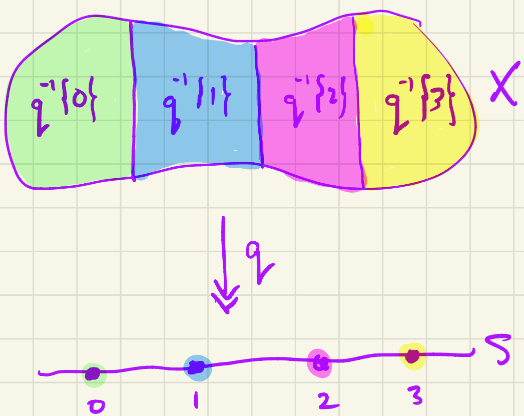
Recall that for X a set, an equivalence relation $\sim$ on X is a reflexive $(x \sim x)$, symmetric $(x \sim y \Rightarrow y \sim x)$, transitive $(x \sim y \sim z \Rightarrow x \sim z)$ relation.

We will write $[x] = [x]_\sim = \{y \in X \mid y \sim x\}$ for the $\sim$-equivalence class of $x \in X$. The $\sim$-equivalence classes partition X and

$$X/\sim \; := \{\sim\text{-equivalence classes}\} = \{[x] \mid x \in X\}.$$

We have a surjection $q : X \to X/\sim$, $x \mapsto [x]$, and $q^{-1}\{[x]\} = [x]$.

Every surjective fn $q : X \twoheadrightarrow S$ gives rise to an equivalence relation $\sim$ on X, where $x \sim y \iff q(x) = q(y)$.



This induces a bijection

$$\begin{aligned} S &\longrightarrow X/\sim \\ s &\longmapsto q^{-1}\{s\} \end{aligned}$$

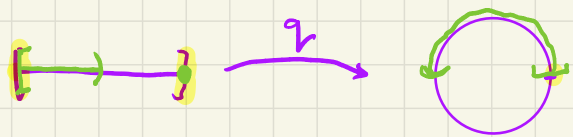
So, for a space X and surjective function $q: X \twoheadrightarrow S$, what topology should we put on S?

① The quotient topology on S has open set $U \subseteq S$ s.t. $q^{-1}U$ is open in X.

② This is the finest topology on S s.t. q is cts.

E.g. (a) $q : [0,1] \longrightarrow [0,1]/\underbrace{0\sim 1}$

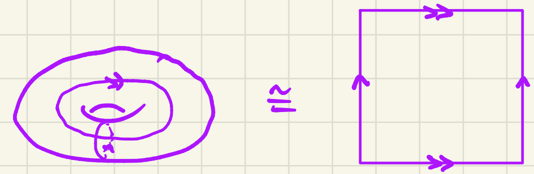
shorthand for "the equivalence relation generated by $0\sim 1$"



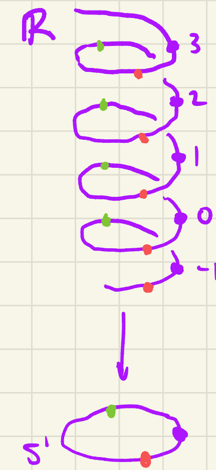
TPS Visualize $\{[x] \mid 0 \leq x < \frac{1}{2}\}$.

Is this open in $[0,1]/0\sim 1$?

(b) $T^2 \cong [0,1]^2 \Big/ \begin{matrix}(0,y)\sim(1,y)\\ (x,0)\sim(x,1)\end{matrix}$



(c) For $x,y \in \mathbb{R}$, define $x\sim y$ iff $x-y\in\mathbb{Z}$. Then $\mathbb{R}/\sim \; = \mathbb{R}/\mathbb{Z}$ is a circle!



(d) For $A \subseteq X$ define $x\sim y$ iff $x=y$ or $x,y\in A$.

Write $X/A := X/\sim$. This is X with "A collapsed to a point."

(i) The cone on X is

$$CX := X\times[0,1] \big/ X\times\{0\}$$