Pf of (c) Let $\mathcal{B}$ be a countable basis of X and $\mathcal{U}$ an open cover of X.

Define $\mathcal{B}' := \{B \in \mathcal{B} \mid B \subseteq U$ for some $U \in \mathcal{U}\}$; it's countable.

For $B \in \mathcal{B}'$, choose $U_B \in \mathcal{U}$ s.t. $B \subseteq U_B$. Then $\mathcal{U}' = \{U_B \mid B \in \mathcal{B}'\} \subseteq \mathcal{U}$ is countable.

WTS $\mathcal{U}'$ covers X. For $x \in X$, know $x \in U_0$ for some $U_0 \in \mathcal{U}$. Since $\mathcal{B}$ is a basis, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U_0$. Thus $B \in \mathcal{B}'$ and $U_B \in \mathcal{U}'$ with $x \in B \subseteq U_B$. This shows $\mathcal{U}'$ is a cover. $\square$

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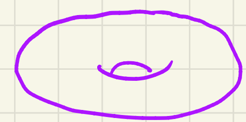
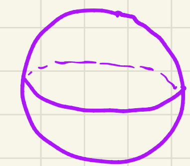
5. X. 22

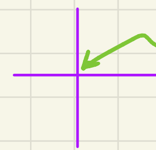
Manifolds A space M is *locally Euclidean of dimension n* when any of the following equivalent conditions holds:

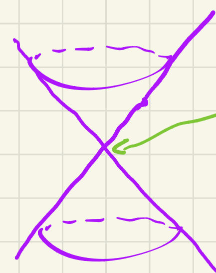
- every pt of M has a nbhd in M homeomorphic to an open subset of $\mathbb{R}^n$
- every pt of M has a nbhd in M homeomorphic to an open ball in $\mathbb{R}^n$
- every pt of M has a nbhd in M homeomorphic to $\mathbb{R}^n$.

An **n-dimensional topological manifold** is a

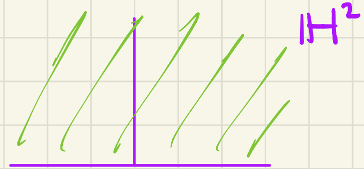
- second countable
- Hausdorff space that is
- locally Euclidean of dimension n.

E.g. $\mathbb{R}^n$, open subsets of manifolds,  , 

Non-e.g. $\{(x,y)\in\mathbb{R}^2 \mid xy=0\}$  not loc Euc here

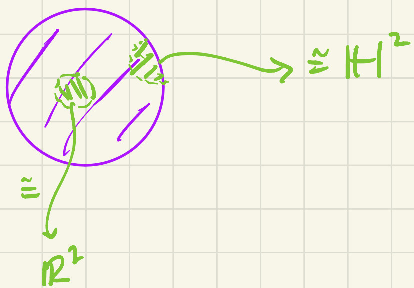
$\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=z^2\}$ 

$\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ is the closed n-dimensional upper half space.



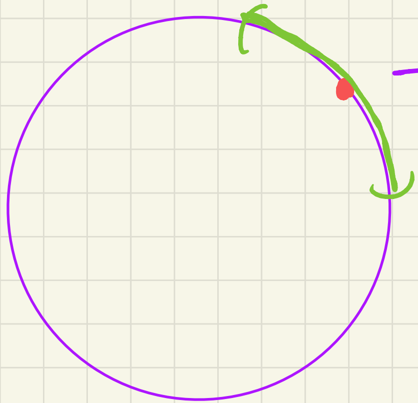
An n-dimensional manifold with boundary is a second countable Hausdorff space in which every pt has a nbhd $\cong \mathbb{R}^n$ or $\mathbb{H}^n$.

E.g.



If p has a nbhd $\cong \mathbb{R}^n$ it's called an interior point. If p has a nbhd $\cong \mathbb{H}^n$ and none $\cong \mathbb{R}^n$ it's a boundary point.

Prop The collection of interior pts is an n-manifold, and $\partial M := \{\text{boundary pts}\}$ is an $(n-1)$-manifold. (b/c $\partial \mathbb{H}^n \cong \mathbb{R}^{n-1}$)

 $\xrightarrow{\cong} \mathbb{R}^1$

so S^1 is a 1-manifold

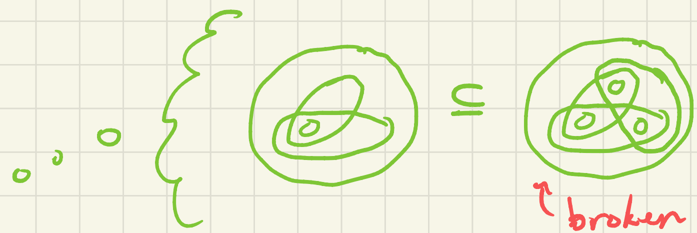
New spaces from old

Goal Put "natural" topologies on subsets, products, quotients, etc.

Three characterizations

1. Classical definition (explicit, by fiat).
2. Coarsest/finest topology such that maps into/out of the space are cts.
3. Universal property (justifies ①)

Defn Given topologies $\mathcal{T} \subseteq \mathcal{T}'$ on X call $\mathcal{T}'$ <u>finer</u> and $\mathcal{T}$ <u>coarser</u>.



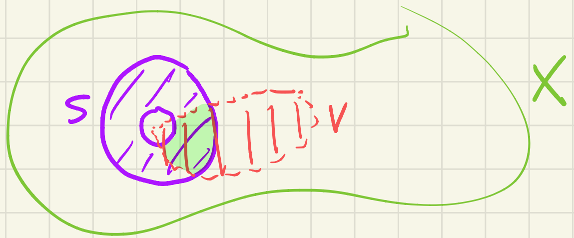
broken into smaller pieces, like finer coffee grounds

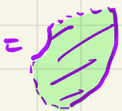
Subspaces

Given $S \subseteq X$ space an arbitrary subset, what topology do we put on it?

① The subspace topology on S is

$$T_S := \{ U \subseteq S \mid U = S \cap V \text{ for some } V \subseteq X \text{ open} \}.$$



$U =$  is open in S

② **Prop** T_S is the coarsest topology on S such that

iota $\iota : S \hookrightarrow X$ is continuous.

Pf Suppose $\iota : (S, T) \hookrightarrow X$ is cts. Then $\forall V \subseteq X$ open,
$\iota^{-1} V = S \cap V \in T$. Thus $T_S \subseteq T$. Now check $T' \subsetneq T$
$\Longrightarrow \iota : (S, T') \hookrightarrow X$ is not cts. $\square$

3 **Thm** For X a space and $S \subseteq X$ a subspace, a map $f : Y \to S$ is cts iff $\iota f : Y \to X$ is cts:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr, "\iota f"'] & S \arrow[d, hook, "\iota"] \arrow[dl, phantom, "\Updownarrow\ \text{cts}"] \\
& X
\end{tikzcd}.$$

Pf ($\Rightarrow$) ι is cts & composition preserves continuity.

($\Leftarrow$) For $U \subseteq S$ open, $U = V \cap S$ for some $V \subseteq X$ open. By continuity of ιf, $(\iota f)^{-1} V = f^{-1}(\iota^{-1} V) = f^{-1} U$ is open. Since U was arbitrary, f is cts. $\square$

Cor For $f : X \to Y$ cts,

$$\begin{tikzcd}
S \arrow[d, hook] \arrow[dr, "f|_S\ \text{cts}"] & & \\
X \arrow[r, "f"'] \arrow[rr, bend left, "\text{cts}"] \arrow[dr, "\text{cts}"'] & Y \arrow[r, hook] & Z \\
& T \arrow[u, hook] &
\end{tikzcd}$$

where the domain of each $\hookrightarrow$ has subspace topology. $\square$

Thm Suppose S is a subset of a space X. The subspace topology on S is the unique topology satisfying the universal property.

by abstract nonsense!

Pf Write S_s for S w/ subspace top, S_g for S w/ some top satisfying the univ prop. Suffices to show $id_S : S_s \to S_g$ is a homeomorphism.

Consider the diagrams

$$\begin{tikzcd} S_s \arrow[r, "id_S"] \arrow[dr, hook, "\iota"'] & S_g \arrow[d, hook, "\iota"'] \\ & X \end{tikzcd} \quad , \quad \begin{tikzcd} S_g \arrow[r, "id_S"] \arrow[dr, hook, "\iota"'] & S_s \arrow[d, hook, "\iota"] \\ & X \end{tikzcd} .$$

By the univ property it suffices to show ι is cts in each case.

We already checked that S_s is the coarsest topology s.t. $\iota : S_s \hookrightarrow X$ is cts. We have that $\iota : S_g \hookrightarrow X$ is cts by applying the univ prop to

$$\begin{tikzcd} S_g \arrow[r, "id"] \arrow[dr, hook, "\iota"'] & S_g \arrow[d, hook, "\iota"] \\ & X \end{tikzcd}$$

, noting that $id : S_g \to S_g$ is necessarily cts.

Hence $id_S : S_s \rightleftarrows S_g : id_S$ are both cts. $\square$

space

For $S \subseteq X$ call a top τ on S a **subspace topology**

subset

when $\forall f : Y \to S$ function,

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr, "if"'] \arrow[dr, phantom, shift left=3, "\Updownarrow \text{cts}"] & S \arrow[d, hook, "i"] \\
& X
\end{tikzcd}$$

Embeddings

$f : A \xrightarrow{\cong} fA$

A cts fn $f: A \to X$ is an embedding when it is a homeomorphism onto its image $fA \subseteq X$ w/ the subspace topology.

Prop A cts injective map that is either open or closed is an embedding.

Pf Reading / exercise. $\square$

Prop Hausdorff, first / second countable properties are preserved under taking subspaces. $\square$

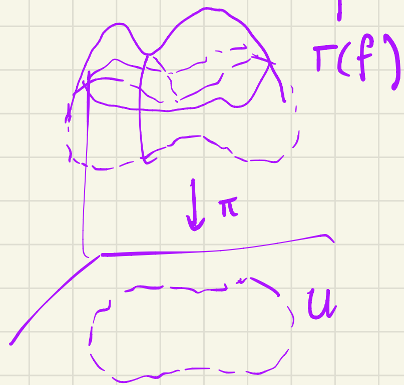
Since $\mathbb{R}^n$ is Hausdorff & 2nd countable, any locally Euclidean space which embeds in some $\mathbb{R}^n$ is a manifold.

(Fact Every manifold embeds in some $\mathbb{R}^n$.) Cf. Whitney embedding

E.g. If $U \subseteq \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^k$ is cts, then the graph of f

$$\Gamma(f) := \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^k$$

is a manifold homeomorphic to U (via projection)



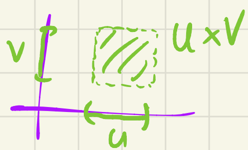
Note Injectivity of π = "vertical line test"

Products

7.X.22

For spaces X, Y, what topology should $X \times Y$ have?

① The product topology on $X \times Y$ is the topology generated by the basis $\mathcal{B} = \{U \times V \mid U \subseteq X \;\&\; V \subseteq Y \text{ open}\}$



Moral Exercise Check that $\mathcal{B}$ is a basis!