3.X.22

Huh? This will appear more natural after discussing bases.

Facts $\forall X, Y, Z$ spaces

(a) Every constant map $X \to Y$ is cts — Continuous

(b) The identity $id_X : X \to X$ is cts

(c) If $f: X \to Y$ is cts, so is $f|_U$ $\forall U \subseteq X$ open — restriction of f to U

(d) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are cts, then so is $g \circ f : X \to Z$.

Even better: $f: X \to Y$ is cts iff $\forall x \in X$ $\exists V_x \subseteq X$ nbhd of x s.t. $f|_{V_x}$ is cts.

Slogan Continuity is a local property.

Note (b), (d) along with associativity of composition say that $\mathrm{Top} := (\{\text{top'l spaces}\}, \{\text{cts maps}\})$ forms a category.

An isomorphism in a category is a map $\varphi : a \to b$ with a two-sided inverse $\varphi^{-1} : b \to a$.

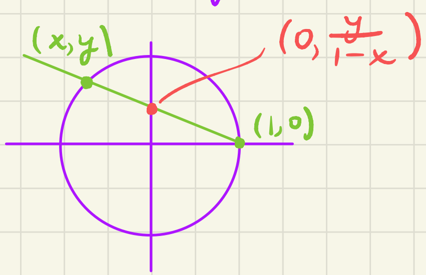
An isomorphism in Top is called a homeomorphism; it's a continuous function $\varphi : X \to Y$ with a continuous inverse $\varphi^{-1} : Y \to X$. Write $\varphi : X \cong Y$.

E.g. (1) Any affine transformation of $\mathbb{R}^n$ is a homeomorphism.

(2) $$F : \mathbb{B}^n \longrightarrow \mathbb{R}^n \qquad x \longmapsto \frac{x}{1-|x|}$$ with inverse $$G : \mathbb{R}^n \longrightarrow \mathbb{B}^n \qquad y \longmapsto \frac{y}{1+|y|}$$

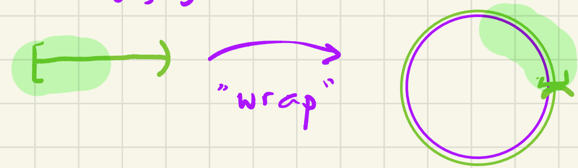
(3) Stereographic projection

$$F : S^n \setminus \{(1,0,\dots,0)\} \xrightarrow{\cong} \mathbb{R}^n$$
$$(x_0, \dots, x_n) \longmapsto \left(\frac{x_1}{1-x_0}, \dots, \frac{x_n}{1-x_0}\right)$$



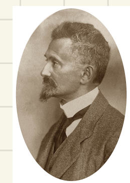
$[0,1) \longrightarrow S^1$

Non-e.g.:



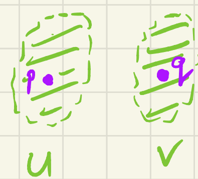
is a continuous bijection but NOT a homeomorphism: its inverse is not continuous

Separation



Felix Hausdorff
1868–1942

A space X is Hausdorff when $\forall\, p \neq q \in X$ $\exists$ nbhds U of p, V of q with $U \cap V = \emptyset$:



In a Hausdorff space, you can separate points w/ open sets

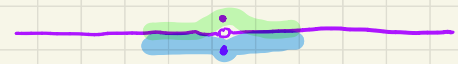
Prop If X is a metric space, then it is Hausdorff.

Pf For $p \neq q \in X$, let $r = d(p,q) > 0$. Then $B(p, \frac{r}{2}) \cap B(q, \frac{r}{2}) = \emptyset$

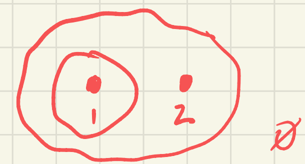
by Δ inequality. $\square$

Non-Hausdorff spaces (1) Trivial topology on X when $|X|>1$.

(2) Spec $\mathbb{C}[t]$ (or nearly any Zariski spectrum)

(3) The "line with two origins" 

which we will formally construct via the quotient topology.

TPS Construct a new non-Hausdorff space. 

TPS Give $\mathcal{N} := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ the metric topology induced by $\mathbb{R}$.

For (x_i) a sequence in a space X, show that

$x_i \longrightarrow x \in X$ iff $f : \mathcal{N} \longrightarrow X$ is continuous.

$$\frac{1}{n} \longmapsto x_n$$

$$0 \longmapsto x$$

Bases

X a space, $\mathcal{B} \subseteq 2^X$ is a basis for the topology of X when

(i) every $B \in \mathcal{B}$ is open

(ii) every open subset of X is a union of elements of $\mathcal{B}$.

E.g. • For M a metric space, $\mathcal{B} = \{B(x,\varepsilon) \mid x \in M, \varepsilon > 0\}$ is a basis

• For X discrete, $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a basis

Prop For X, Y spaces, $\mathcal{B}$ a basis for X, $\mathcal{C}$ a basis for Y,
$f : X \longrightarrow Y$ is cts iff $\forall y \in Y$ and $x \in X$ s.t. $f(x) = y$,
if $y \in C \in \mathcal{C}$ then $\exists x \in B \in \mathcal{B}$ s.t. $fB \subseteq C$.

Cor A function between metric spaces $f : X \longrightarrow Y$ is cts iff $\forall y \in Y$
and $x \in X$ s.t. $f(x) = y$, $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $fB(x,\delta) \subseteq B(y,\varepsilon)$. $\square$

Pf of Prop Suppose f cts, $f(x)=y$, and $y \in C \in \mathcal{C}$. Then $f^{-1}C$ is open so $x \in f^{-1}C = \bigcup_{B \in I} B$ for some $I \subseteq \mathcal{B}$. Thus x is in one of these $B \implies fB \subseteq C$.

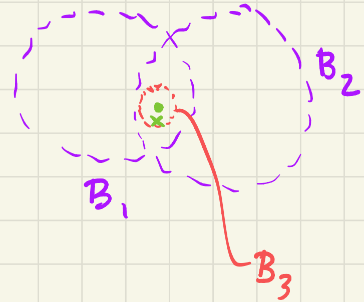
For the converse, if $U \subseteq Y$ is open then $U = \bigcup_{C \in J} C$ for some $J \subseteq \mathcal{C}$. We have $f^{-1}U = \bigcup_{C \in J} f^{-1}C$ so suffices to show $f^{-1}C$ is open. This is the case if for each $x \in f^{-1}C$, $x \in B_x \subseteq f^{-1}C$ for some $B_x \in \mathcal{B}$. (Indeed, then $f^{-1}C = \bigcup_{x \in f^{-1}C} B_x$.) But such B_x is exactly what the hypotheses guarantee! $\square$

Note See Prop 2.43 for a similar result.

Prop $\mathcal{B} \subseteq 2^X$ is a basis for some topology on X iff

(i) $\bigcup_{B \in \mathcal{B}} B = X$ (ii) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$

Pf Reading (2.44). $\square$



(ii) $\Rightarrow$ $U \cap V$ open for U, V open

Countability

- For $p \in X$, a collection $\mathcal{B}_p \subseteq 2^X$ of nbhds of p is a neighborhood basis for X at p if every nbhd of p contains some $B \in \mathcal{B}_p$.
- Call X first countable if $\exists$ countable nbhd basis at each point of X.

E.g. If X is a metric space, $\mathcal{B}_p := \{B(p, \varepsilon) \mid \varepsilon \in \mathbb{Q}_{>0}\}$ is a countable nbhd basis at $p \in X$. Hence X is first countable.

Sequence Lemma Suppose X is first countable, $A \subseteq X$, $x \in X$.

(a) $x \in \bar{A} \iff x =$ limit of pts in A

(b) $x \in A^\circ \iff (x_i \to x \Rightarrow x_i \in A$ for $i \gg 0)$

Call X second countable when it admits a countable basis.

E.g. $\{B(x,\varepsilon) \mid x \in \mathbb{Q}^n, \varepsilon \in \mathbb{Q}_{>0}\}$ is a countable basis of $\mathbb{R}^n$, so Euclidean space is second countable.

Non-e.g. The long line.

An open cover of a space X is a collection of open sets $\mathcal{U}$ s.t. $X = \bigcup_{u \in \mathcal{U}} u$.

A subcover of $\mathcal{U}$ is $\mathcal{U}' \subseteq \mathcal{U}$ that still covers.

Thm If X is second countable, then

(a) X is first countable,

(b) X is separable (contains a countable dense subset),

(c) X is Lindelöf (every open cover has a countable subcover).

Pf of (c) Let $\mathcal{B}$ be a countable basis of X and $\mathcal{U}$ an open cover of X.
Define $\mathcal{B}' := \{B \in \mathcal{B} \mid B \subseteq U$ for some $U \in \mathcal{U}\}$; it's countable.
For $B \in \mathcal{B}'$, choose $U_B \in \mathcal{U}$ s.t. $B \subseteq U_B$. Then $\mathcal{U}' = \{U_B \mid B \in \mathcal{B}'\} \subseteq \mathcal{U}$ is countable.

WTS $\mathcal{U}'$ covers X. For $x \in X$, know $x \in U_0$ for some $U_0 \in \mathcal{U}$. Since $\mathcal{B}$ is a basis, $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U_0$. Thus $B \in \mathcal{B}'$ and $U_B \in \mathcal{U}'$ with $x \in B \subseteq U_B$. This shows $\mathcal{U}'$ is a cover. $\square$

5. X. 22

Manifolds A space M is locally Euclidean of dimension n when any of the following equivalent conditions holds:

- every pt of M has a nbhd in M homeomorphic to an open subset of $\mathbb{R}^n$
- " " open ball in $\mathbb{R}^n$
- " " $\mathbb{R}^n$.