30. IX. 22

Discussion

(1) Assessment & self-assessment

Goal: assess competence w/ learning objectives

required
- weekly HW + revisions
- take-home midterm + revision
- final oral exam

- Alternative/supplemental:
  - paper/presentation (potentially collaborative)
  - presenting HW sol'ns

Q Weighting of objectives?

(2) Joint expectations

Make space $\forall$ to contribute

$\exists$ differences

Listen

Compassionate communication

Checking in w/ partners

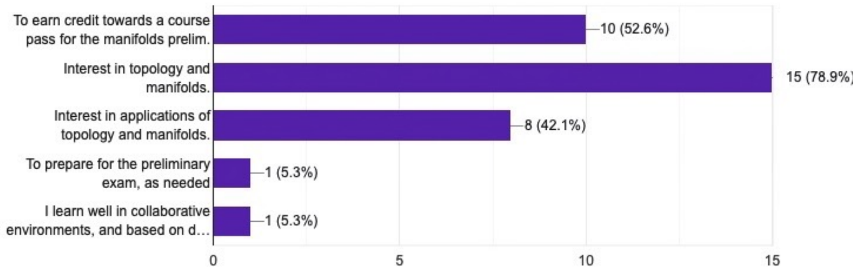
Respect

Be present & ready collaborate

Encourage mathematical risk-taking/vulnerability

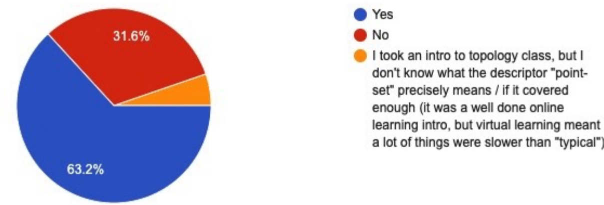
Celebrate mistakes

Welcome Survey



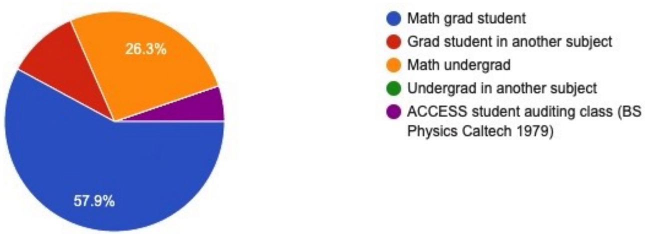
Have you taken a point-set topology course previously?

19 responses



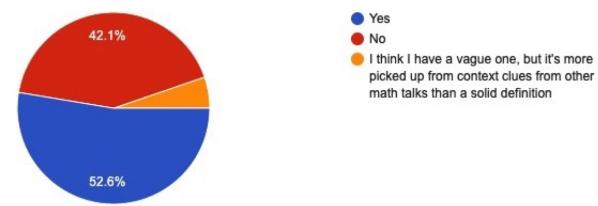
What kind of student are you?

19 responses



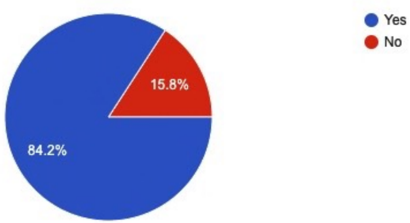
Do you feel like you have an intuitive sense of what a (topological or smooth) manifold is?

19 responses



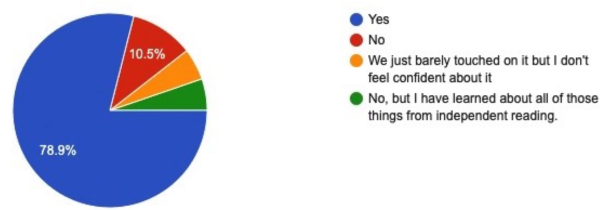
Have you previously seen the definition of a topology (in terms of a set and open subsets satisfying axioms) in a course setting?

19 responses



Have you taken an algebra course that included the study of abstract groups (including normal subgroups, quotients, and group actions)?

19 responses



Problem session Tuesdays 13:00-14:00

Kyle's drop-in hours Thursdays 15:00-16:00

Q & A

Internet: Don't hunt for sol'ns; no Internet on the midterm

Final: Converse with me about topology problems

Differential geometry: Riemannian metrics start in 546

Topological Spaces

X a set, $2^X := \{A \mid A \subseteq X\}$ its power set

A topology on X is $\mathcal{T} \subseteq 2^X$ s.t.

(i) $X, \emptyset \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under pairwise (and hence finite) intersections:

$$U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}.$$

(iii) $\mathcal{T}$ is closed under arbitrary unions:

$$\{U_\alpha \mid \alpha \in S\} \subseteq \mathcal{T} \implies \bigcup_{\alpha \in S} U_\alpha \in \mathcal{T}.$$

- $(X, \mathcal{T})$ is a topological space
- $U \in \mathcal{T}$ called open subsets of X
- $x \in X$ a point of X

- For $U \in \mathcal{T}$, $X \setminus U$ called closed
- $p \in U \in \mathcal{T}$ neighborhood of p

E.g.: (1) $(X, 2^X) =:$ discrete topology

(2) $(X, \{\emptyset, X\}) =:$ trivial topology

(3) For (M, d) a metric space, unions of open balls $B(x, \varepsilon) = \{ y \in M \mid d(x,y) < \varepsilon \}$ form the metric topology on M.

(a) Euclidean topology on $\mathbb{R}^n$

(b) Similarly for unit interval $I = [0,1]$,

open unit ball $\mathbb{B}^n := \{ x \in \mathbb{R}^n \mid |x| < 1 \} \subseteq \mathbb{R}^n$,

closed unit ball $\bar{\mathbb{B}}^n := \{ x \in \mathbb{R}^n \mid |x| \leq 1 \} \subseteq \mathbb{R}^n$,

unit circle $S^1 := \{ x \in \mathbb{R}^2 \mid |x| = 1 \} \subseteq \mathbb{R}^2$,

unit sphere $S^n := \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \} \subseteq \mathbb{R}^{n+1}$.

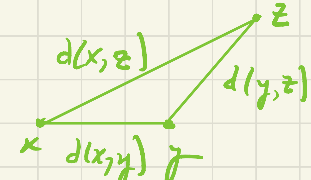
$d : M \times M \longrightarrow \mathbb{R}$ s.t. $\forall x, y, z \in M$,

(1) $d(x,x) = 0$

(2) $x \neq y \Rightarrow d(x,y) > 0$

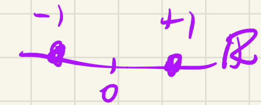
(3) $d(x,y) = d(y,x)$

(4) $d(x,z) \leq d(x,y) + d(y,z)$



TPS (Think-Pair-Share) Describe $\mathbb{R}^0$ and S^0.

$$\mathbb{R}^0 = \{0\} \qquad S^0 = \{-1, 1\}$$



E.g. (4) $U \subseteq X$ open in the cofinite topology iff $|X \setminus U| < \infty$ or $U = \emptyset$.

(5) For a commutative unital ring R, $\operatorname{Spec} R := \{ p \subseteq R \mid p \text{ prime ideal} \}$.

Define Zariski closed subsets of R to be, for $I \subseteq R$ an ideal,

$$V(I) := \{ p \in \operatorname{Spec} R \mid p \supseteq I \}.$$

Note Can specify a topology on X by closed sets as long as

- X, $\emptyset$ closed
- finite unions of closed sets are closed
- arbitrary intersections of closed sets are closed.

(Hint: De Morgan's Laws.)

Now leaving $\mathcal{T}$ implicit.

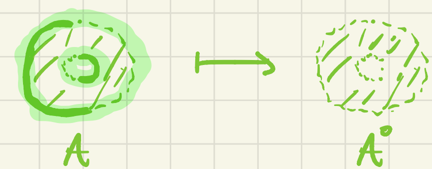
Fix a topological space X.

For $A \subseteq X$, the <u>closure</u> of A in X is

$$\bar{A} := \bigcap_{\substack{A \subseteq B \subseteq X \\ B \text{ closed}}} B = \text{smallest closed set containing } A.$$

The <u>interior</u> of A in X is

$$\operatorname{Int} A = A^\circ := \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U = \text{largest open set contained by } A$$



The exterior of A is $\operatorname{Ext} A := X \setminus \bar{A}$ and its boundary is

$$\partial A := X \setminus (A^\circ \cup \operatorname{Ext} A)$$

See Prop 2.8 for important properties of $\overline{(\ )}$, $(\ )^\circ$, Ext, ∂.

Aside You can also specify a topology on X via a closure operator $c: 2^X \longrightarrow 2^X$ satisfying

(1) $c(\emptyset) = \emptyset$

(2) $\forall A \subseteq X$, $A \subseteq c(A)$

(3) $\forall A \subseteq X$, $c(A) = c(c(A))$

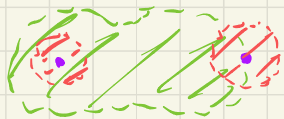
(4) $\forall A, B \subseteq X$, $c(A \cup B) = c(A) \cup c(B)$.

c tells you what A can "see"

Logic of a topological space? Interior = necessity, closure = possibility models "S4 modal logic".

- For $A \subseteq X$, $p \in X$ is a limit (or accumulation or cluster) point of A when every neighborhood of p contains a point of A other than p: $\forall\, U \subseteq X$ open with $p \in U$, $U \cap (A \setminus \{p\}) \neq \emptyset$.

E.g.



, 0 is a limit point of $H = \{ \frac{1}{n} \mid n \in \mathbb{Z}_{\geq 1} \} \subseteq \mathbb{R}$ but $\frac{1}{2}$ is not.

- A point $p \in X$ is an isolated point of A if $\exists$ nbhd U of p s.t. $A \cap U = \{p\}$.

E.g.

isolated, $\frac{1}{n}$ is an isolated point of H $\forall n \geq 1$.

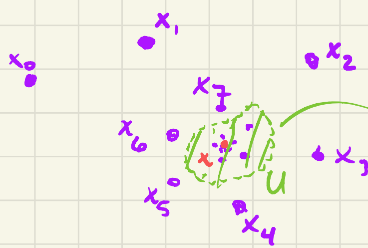
- A subset $A \subseteq X$ is dense in X when $\overline{A} = X$.

E.g. $\mathbb{Q} \subseteq \mathbb{R}$.

Probing spaces with sequences and maps

A sequence $(x_i)_{i\in\mathbb{N}}$ in X converges to $x \in X$ when

$\forall$ nbhd U of x, $\exists N \in \mathbb{N}$ s.t. $\forall i \geq N$, $x_i \in U$.



$N=8$ works for this U

Write $x_i \longrightarrow x$ or

$$\lim_{i\to\infty} x_i = x.$$

A function $f: X \to Y$ between topological spaces is continuous

when $\forall\, U \subseteq Y$ open, $f^{-1}U := \{x \in X \mid f(x) \in U\} \subseteq X$ is open.