MATH 544: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 10

Problems taken from *Introduction to Topological Manifolds* are marked ITM x–y. Please review the syllabus for expectations and policies regarding homework. I also **strongly recommend** that you think about how to do problems 8–5, 8–11, and 9–5.

Problem 1 (ITM 8–1). (a) Suppose $U \subseteq \mathbb{R}^2$ is an open subset and $x \in U$. Show that $U \setminus \{x\}$ is not simply connected.

(b) Show that if $n > 2$, then $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}^2$.

Problem 2 (ITM 8–2, INVARIANCE OF DIMENSION, 2-DIMENSIONAL CASE). Prove that a nonempty topological space cannot be both a 2-manifold and an n-manifold for some $n > 2$.

Problem 3 (ITM 8–3, INVARIANCE OF BOUNDARY, 2-DIMENSIONAL CASE). Suppose M is a 2-dimensional manifold with boundary. Show that a point of M cannot be both a boundary point and an interior point.

Problem 4 (ITM 9–2). The *center* of a group G is the subgroup

$$ZG := \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

of elements of G commuting with every element of G. (Moral exercise: check that ZG really is a subgroup.) Show that a free group on two or more generators has trivial center.

Problem 5 (ITM 9–4). Let G_1, G_2, H_1, H_2 be groups and let $f_i \colon G_i \to H_i$ be homomorphisms for $i = 1, 2$.

(a) Show that there exists a unique homomorphism $f_1 * f_2 \colon G_1 * G_2 \to H_1 * H_2$ such that the following diagram commutes for $i = 1, 2$:

$$\begin{array}{ccc} G_1 * G_2 & \xrightarrow{f_1 * f_2} & H_1 * H_2 \\ \uparrow{\scriptstyle \iota_i} & & \uparrow{\scriptstyle \iota_i'} \\ G_i & \xrightarrow[f_i]{} & H_i, \end{array}$$

where $\iota_i \colon G_i \to G_1 * G_2$ and $\iota_i' \colon H_i \to H_1 * H_2$ are the canonical injections.

(b) Let S_1, S_2 be disjoint sets, and let R_i be a subset of the free group $F(S_i)$ for $i = 1, 2$. Prove that $\langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$ is a presentation of the free product group $\langle S_1 \mid R_1 \rangle * \langle S_2 \mid R_2 \rangle$.