MATH 544: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 7

Problems taken from *Introduction to Topological Manifolds* are marked ITM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ITM 6–2). Note that both a disk and a Möbius band are manifolds with boundary, and both boundaries are homeomorphic to S^1. Show that $\mathbb{RP}^2$ is homeomorphic to a space obtained by attaching a disk to a Möbius band along their boundaries.

Problem 2 (ITM 6–3). Show that the Klein bottle is homeomorphic to a quotient obtained by attaching two Möbius bands together along their boundaries.

Problem 3 (ITM 6–6). For each of the following surface presentations, compute the Euler characteristic and determine which of the standard surfaces it represents:

(a) $\langle a, b, c \mid abacb^{-1}c^{-1}\rangle$,
(b) $\langle a, b, c \mid abca^{-1}b^{-1}c^{-1}\rangle$,
(c) $\langle a, b, c, d, e, f \mid abc,\ bde,\ c^{-1}df,\ e^{-1}fa\rangle$,
(d) $\langle a, b, c, d, e, f, g, h, i, j, k, \ell, m, n, o \mid$
$abc,\ bde,\ dfg,\ fhi,\ haj,\ c^{-1}k\ell,\ e^{-1}mn,\ g^{-1}ok^{-1},\ i^{-1}\ell^{-1}m^{-1},\ j^{-1}n^{-1}o^{-1}\rangle$.

Problem 4 (ITM 7–3). Let X be a path-connected topological space, and let $p, q \in X$. Show that all paths from p to q give the same isomorphism of $\pi_1(X, p)$ with $\pi_1(X, q)$ if and only if $\pi_1(X, p)$ is Abelian.

Problem 5 (ITM 7–5). Let G be a topological group.

(a) Prove that up to isomorphism, $\pi_1(G, g)$ is independent of the choice of base point $g \in G$.
(b) Prove that $\pi_1(G, g)$ is Abelian. [*Hint*: if f, g are loops based at $1 \in G$, apply the square lemma to $F\colon I \times I \to G$ given by $F(s, t) = f(s)g(t)$.]