MATH 544: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 4

Problems taken from *Introduction to Topological Manifolds* are marked ITM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ITM 4–1,2). Show that for $n > 1$, $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}$. [*Hint*: if $U \subseteq \mathbb{R}$ is open and $x \in U$, then $U \smallsetminus \{x\}$ is not connected.] Use this to prove that a nonempty topological space cannot be both a 1-manifold and an n-manifold for some $n > 1$.

Problem 2 (ITM 4–15). Suppose that G is a topological group.

(a) Show that every open subgroup of G is also closed.
(b) For any neighborhood U of 1, show that the subgroup $\langle U \rangle$ generated by U is open and closed in G.
(c) For any connected subset $U \subseteq G$ containing 1, show that $\langle U \rangle$ is connected.
(d) Show that if G is connected, then every connected neighborhood of 1 generates G.

Problem 3 (ITM 4–18). Let M_1 and M_2 be n-manifolds. For $i = 1, 2$, let $B_i \subseteq M_i$ be regular coordinate balls, and let $M_i' = M_i \smallsetminus B_i$. Choose a homeomorphism $f\colon \partial M_2' \to \partial M_1'$. (You may assume that $\partial M_i' \cong S^{n-1}$ for $i = 1, 2$ and thus such an f exists.) Let $M_1 \# M_2$ (called the *connected sum* of M_1 and M_2) be the adjunction space $M_1' \cup_f M_2'$.

(a) Show that $M_1 \# M_2$ is an n-manifold (without boundary).
(b) Show that if M_1 and M_2 are connected and $n > 1$, then $M_1 \# M_2$ is connected. (Is this also true for $n = 1$?)
(c) Show that if M_1 and M_2 are compact, then $M_1 \# M_2$ is compact.

Problem 4 (ITM 4–13). Let X be a locally compact Hausdorff space. The *one-point compactification* of X is the topological space X^* defined as follows: For ∞ some object not in X, let $X^* = X \amalg \{\infty\}$ with the following topology $\mathscr{T}$ consisting of open subsets of X along with $U \subseteq X^*$ such that $X^* \smallsetminus U$ is a compact subset of X.

(a) Show that $\mathscr{T}$ is a topology.
(b) Show that X^* is a compact Hausdorff space.
(c) Use stereographic projection to show that $S^n \cong (\mathbb{R}^n)^*$.
(d) Show that X is open in X^* and has the subspace topology.
(e) Show that X is dense in X^* if and only if X is noncompact.

Problem 5. Let X be a metric space with metric d. Given $x \in X$ and nonempty $A \subseteq X$, define

$$d(x, A) := \inf\{d(x, a) \mid a \in A\}.$$

Give a direct proof of Urysohn's lemma for metric spaces by considering the function

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$