MATH 544: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 3

Problems taken from *Introduction to Topological Manifolds* are marked ITM x–y. Please review the syllabus for expectations and policies regarding homework. In addition to these problems, I recommend working out ITM 3–22 and 3–23 as well.

Problem 1 (ITM 3–13). Suppose X and Y are topological spaces and $f\colon X \to Y$ is a continuous map. Prove the following:

(a) If f admits a continuous left inverse, it is a topological embedding.
(b) If f admits a continuous right inverse, it is a quotient map.
(c) Give examples of a topological embedding with no continuous left inverse, and a quotient map with no continuous right inverse.

Problem 2 (ITM 3–14). Show that real projective space $\mathbb{RP}^n$ is an n-manifold. [*Hint*: Consider the subsets $U_i \subseteq \mathbb{R}^{n+1} \smallsetminus \{0\}$ with $x_i = 1$.]

Problem 3. Let H^n denote the upper n-hemisphere

$$H^n := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} > 0\}$$

and let $\bar{H}^n$ denote its closure.

(a) Prove that H^n is homeomorphic to $\mathbb{R}^n$.
(b) Prove that $\mathbb{RP}^n$ is homeomorphic to $\bar{H}^n/{\sim}$ where $\sim$ is the equivalence relation identifying antipodal points on $\partial\bar{H}^n \cong S^{n-1}$.
(c) Conclude that $\mathbb{RP}^n \cong \mathbb{R}^n \cup \mathbb{RP}^{n-1}$ and thus by induction

$$\mathbb{RP}^n \cong \mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \cdots \cup \mathbb{R}^1 \cup \mathbb{R}^0.$$

Problem 4 (ITM 3–16). Let X be the subset $(\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2$. Define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ for $x \neq 0$. Show that the quotient space $X/{\sim}$ is locally Euclidean and second countable but not Hausdorff. (This space is called the *line with two origins*.)

Problem 5 (ITM 3–24). Consider the action of $\mathrm{O}(n)$ on $\mathbb{R}^n$ by matrix multiplication as in Example 3.88(b). Prove that the quotient space $\mathbb{R}^n/\mathrm{O}(n)$ is homeomorphic to $[0, \infty)$.