MATH 544: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 2

Problems taken from *Introduction to Topological Manifolds* are marked ITM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ITM 2–5). For each of the following properties, give an example of consisting of two subsets $X, Y \subseteq \mathbb{R}^2$, both considered as subspaces of the Euclidean plane, together with a map $f\colon X \to Y$ that has the indicated property. (No justifications required.)

(a) The map f is open but neither closed nor continuous.
(b) The map f is closed but neither open nor continuous.
(c) The map f is continuous but neither open nor closed.
(d) The map f is continuous and open but not closed.
(e) The map f is continuous and closed but not open.
(f) The map f is open and closed but not continuous.

Problem 2 (ITM 2–7). Prove Proposition 2.39: In a Hausdorff space, every neighborhood of a limit point contains infinitely many points of the set.

Problem 3 (ITM 2–10). Suppose $f, g\colon X \to Y$ are continuous maps and Y is Hausdorff. Show that the *equalizer* of f and g,

$$\operatorname{eq}(f, g) := \{x \in X \mid f(x) = g(x)\},$$

is closed in X. Give a counterexample if Y is not Hausdorff.

Problem 4 (ITM 2–14). Prove Lemma 2.48: Suppose X is a first countable space, A is any subset of X, and x is any point of X.

(a) A point $x \in \overline{A}$ if and only if x is a limit of a sequence of points in A.
(b) A point $x \in \operatorname{Int} A$ if and only if every sequence in X converging to x is eventually in A.
(c) The set A is closed in X if and only if A contains every limit of every convergent sequence of points in A.
(d) The set A is open in X if and only if every sequence in X converging to a point in A is eventually in A.

Problem 5 (ITM 2–19). Let X be a topological space and let $\mathscr{U}$ be an open cover of X.

(a) Suppose we are given a basis for each $U \in \mathscr{U}$ (when considered as a topological space in its own right via the subspace topology). Show that the union of all these bases is a basis for X.
(b) Show that if $\mathscr{U}$ is countable and each $U \in \mathscr{U}$ is second countable, then X is second countable.