MATH 544: TOPOLOGY
FINAL EXAM PRACTICE PROBLEMS

Use these problems to prepare for your final oral exam. You are welcome to collaborate on them. I will ask you about at least one of these problems during your oral exam.

Problem 1. A topological space is called *σ-compact* if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

Problem 2. Prove that the surfaces with polygonal presentations

$$\langle a, b, c, d \mid ad^{-1}cdc^{-1}ba^{-1}b^{-1} \rangle$$

and

$$\langle e, f, g, h \mid ehgfe^{-1}h^{-1}g^{-1}f^{-1} \rangle$$

are homeomorphic.

Problem 3. Recall that $\pi_1(X, p)$ may be regarded as the collection of based homotopy classes of pointed maps $(S^1, 1) \to (X, p)$. Let $[S^1, X]$ denote the set of homotopy classes of maps $S^1 \to X$ (with no conditions on basepoints). Let $\Phi\colon \pi_1(X, p) \to [S^1, X]$ denote the function that forgets basepoints. Show that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, p)$. Conclude that when X is path connected, Φ induces a bijection between $[S^1, X]$ and conjugacy classes in $\pi_1(X, p)$.

Problem 4. Let X denote the union of n lines through the origin in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \smallsetminus X)$.

Problem 5. Let $n, s \in S^2$ denote the north and south poles of S^2 and let $X := S^2/\{n, s\}$. Give X the structure of a CW-complex and use this to compute $\pi_1 X$.