MATH 411: TOPICS IN ADVANCED ANALYSIS
HOMEWORK DUE WEDNESDAY WEEK 9

Problem 1. Suppose that A is an LCA group. Recall that the topology on $\hat{A}$ has subbasis consisting of sets

$$P(K,U) = \{\chi \in \hat{A} \mid \chi(K) \subseteq U\}$$

where $K \subseteq A$ is compact and $U \subseteq S^1$ is open. Let $U = e^{2\pi i(-1/4,1/4)}$ and show that for K any compact neighborhood of $e \in A$, the set $\overline{P(K,U)}$ is a compact neighborhood of $1 \in \hat{A}$. (It follows that $\hat{A}$ is locally compact, but you don't need to prove this here.)

Problem 2. Fix a prime p and give the Prüfer p-group

$$\mu_{p^\infty} := \{z \in S^1 \mid z^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$$

the subspace topology induced by the standard topology on S^1. We will write $\mu_{p^\infty}^{\text{sub}}$ for this topological group to remind ourselves that this not the discrete topology with which we normally endow μ_{p^∞}.

(a) Prove that $\mu_{p^\infty}^{\text{sub}}$ is a dense subspace of S^1.

(b) Use (a) to prove that $\widehat{\mu_{p^\infty}^{\text{sub}}} \cong \mathbb{Z}$.

(c) We have observed that $\hat{\mathbb{Z}} \cong S^1$, so we now know that

$$\widehat{\widehat{\mu_{p^\infty}^{\text{sub}}}} \cong S^1$$

which is not isomorphic to $\mu_{p^\infty}^{\text{sub}}$. Why does this not contradict Pontryagin duality?

Problem 3. For $j \in \mathbb{N}$, let A_j be a nontrivial compact Abelian group, and let $A = \prod_{j\in\mathbb{N}} A_j$. Suppose the topology on A_j is induced a metric d_j such that A_j has diameter 1. Define

$$\begin{aligned} d\colon A \times A &\longrightarrow \mathbb{R}_{\geq 0} \\ (x,y) &\longmapsto \sum_{j\in\mathbb{N}} \frac{d_j(x_j,y_j)}{2^j}. \end{aligned}$$

(a) Show that $d(x,y)$ defines a metric on A that makes A a compact LCA group.

(b) Show that for each $j \in \mathbb{N}$, the projection $A \to A_j$ is a continuous group homomorphism.

(c) Show that if each A_j is finite, then every continuous group homomorphism $\mathbb{R} \to A$ is trivial.

Problem 4. For $j \in \mathbb{N}$, let A_j be a compact Abelian group. Suppose that for $i < j$ there is a continuous group homomorphism $\varphi_i^j\colon A_j \to A_i$. Suppose that for $i < k < j$,

$$\varphi_i^k \circ \varphi_k^j = \varphi_i^j.$$

Let $\lim_j A_j$ be the set of all $x \in \prod_j A_j$ such that for all $i < j$, $\varphi_i^j(x_j) = x_i$.

(a) Show that $\lim A_j$ is a closed subgroup of $\prod_j A_j$. This group is called the *limit* of the A_j.

(b) Show that the projections induce continuous group homomorphisms

$$p_i\colon \lim A_j \longrightarrow A_i$$

for $i \in \mathbb{N}$ that satisfy $\varphi_i^k \circ p_k = p_i$ whenever $k > i$.

(c) Suppose there is a compact Abelian group A and a sequence of continuous group homomorphism $q_i\colon A \to A_i$ such that $\varphi_i^k \circ q_k = q_i$ for all $k > i$. Show there is a unique continuous group homomorphism $\alpha\colon A \to \lim_j A_j$ such that for each i,

$$p_i \circ \alpha = q_i.$$

This is the *universal property* of $\lim_j A_j$.

Problem 5. Read Exercise 5.22 from Deitmar's book or Section 3.1 of Riehl's *Category theory in context* to familiarize yourself with *colimits* (which Deitmar calls *direct limits* and denotes $\varinjlim$). Let (A_j, φ_i^j) be a directed system of compact Abelian groups, and let (B_j, ψ_i^j) be a codirected system of discrete Abelian groups. Prove one (or both) of the following statements.

(a)

$$\widehat{\lim A_j} \cong \operatorname{colim} \widehat{A_j}$$

(b)

$$\widehat{\operatorname{colim} B_j} \cong \lim_j \widehat{B_j}$$