MATH 411: TOPICS IN ADVANCED ANALYSIS
HOMEWORK DUE WEDNESDAY WEEK 5

Problem 1. Suppose $u\colon S^1 \times \mathbb{R}_{>0} \to \mathbb{R}$ is a solution to the heat equation on the circle with $u(x,t) \xrightarrow{t\to 0^+} f(x) \in L^2(S^1)$.

(a) Prove that for each $t > 0$ there is a *bounded* linear operator

$$T(t)\colon L^2(S^1) \longrightarrow L^2(S^1)$$

satisfying

$$u(-,t) = T(t)f.$$

(b) Use the relationship between convolution and Fourier coefficients that you established on a previous homework to deduce that for each $t > 0$ there is a function $g^t \in C^\infty(S^1)$ such that

$$T(t)f = g^t * f.$$

You should include a formula for the Fourier series of g^t.

(c) Prove that for $s, t > 0$,

$$T(s)T(t) = T(s+t).$$

(A family of operators with this property is called a *one-parameter semi-group*.)

(d) [optional] Explain a sense in which it might be reasonable to write

$$T(t) = e^{t\frac{\partial^2}{\partial x^2}}.$$

Problem 2. Suppose u is a solution to the heat equation on the circle with initial condition f. Define

$$\begin{aligned} v\colon S^1 \times \mathbb{R}_{>0} &\longrightarrow \mathbb{R} \\ (x,t) &\longmapsto u(x,-t). \end{aligned}$$

(a) Show that v does not satisfy the heat equation in general, but does satisfy a closely related equation.

(b) [related but different] Can the heat equation be solved "backward in time"? Make sense of what this might mean formally and say something insightful about the class of initial conditions that admit such time-reversed solutions.

Problem 3. Read Section 11.4 of Hsu's book, *Fourier series, Fourier transforms, and function spaces.* Prove Theorem 11.4.4:

The set $\mathscr{B}_{\text{odd}} = \{\sin(2\pi n x) \mid n \geq 1\}$ is an eigenbasis for Δ with domain

$$\mathscr{D}(\Delta)_{\text{Dir}} = \left\{ f \in C^2([0,1/2]) \mid f(0) = f(1/2) = 0 \right\},$$

and the set $\mathscr{B}_{\text{even}} = \{\cos(2\pi n x) \mid n \geq 0\}$ is an eigenbasis for Δ with domain

$$\mathscr{D}(\Delta)_{\text{Neu}} = \left\{ f \in C^2([0,1/2]) \mid f'(0) = f'(1/2) = 0 \right\}.$$

Problem 4 (continuation of Problem 3). We will now compute some explicit solutions to the heat equation with Dirichlet and Neumann boundary conditions.

(a) Compute the sine series of the odd extension of $f(x) = 1$ for $x \in [0,1/2]$ and use this to express the solution to the heat equation with Dirichlet boundary conditions and initial condition f.

(b) Compute the cosine series of the even extension of $f(x) = x$ for $x \in [0,1/2]$ and use this to express the solution to the heat equation with Neumann boundary conditions and initial condition f.