MATH 411: TOPICS IN ADVANCED ANALYSIS
HOMEWORK DUE WEDNESDAY WEEK 3

Problem 1. Let D_n denote the n-th Dirichlet kernel, and let F_n denote the n-th Fejér kernel. Prove that for all n,

$$\int_{-1/2}^{1/2} D_n(x)\,dx = 1$$

and use this to deduce that

$$\int_{-1/2}^{1/2} F_n(x)\,dx = 1.$$

Problem 2. Again let F_n denote the n-th Fejér kernel. Fix some δ such that $0 < \delta < 1/2$.

(a) Prove that for all n and for $\delta \leq |x| \leq 1/2$,

$$F_n(x) \leq \frac{1}{n}\frac{1}{\sin^2(\pi\delta)}.$$

(b) Prove that

$$\lim_{n\to\infty} \int_{\delta\leq|x|\leq 1/2} F_n(x)\,dx = 0.$$

(c) Write "This concludes the proof that the Fejér kernel is a Dirac kernel."

Problem 3. Prove that for $f, g \in C^0(S^1)$,

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

Problem 4. Let $f\colon S^1 \to \mathbb{R}$ be the function defined by

$$f(x) = \sum_{n\geq 1} \frac{\sin 2\pi n x}{2^n} = \frac{\sin 2\pi x}{2} + \frac{\sin 4\pi x}{4} + \frac{\sin 6\pi x}{8} + \frac{\sin 8\pi x}{16} + \cdots .$$

(a) Verify that f is a well-defined function.

(b) Evaluate

$$\int_0^1 f(x)\sin(6\pi x)\,dx.$$

(c) Evaluate

$$\int_0^1 f(x)^2\,dx.$$

Problem 5. For $f \in C^1(S^1)$, prove that

$$\widehat{f'}(n) = (2\pi i n)\hat{f}(n).$$

Problem 6. For $k \in \mathbb{N}$ let $x^k \in L^2(S^1)$ denote the function taking values the usual power x^k for $-1/2 < x \leq 1/2$, made 1-periodic.

(a) Show that

$$\widehat{x^k}(0) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{1}{2^k(k+1)} & \text{if } k \text{ is even.} \end{cases}$$

(b) Show that for $n \neq 0$, $\widehat{x^0}(n) = 0$.

(c) Use integration by parts to determine an inductive formula for $\widehat{x^{k+1}}(n)$.

(d) Give a non-inductive formula for $\widehat{x^k}(n)$.