# MATH 411: TOPICS IN ADVANCED ANALYSIS
# HOMEWORK DUE WEDNESDAY WEEK 2

Make sure to review the homework instructions in the syllabus before writing your solutions. In particular, show your work and write in complete sentences (but also aim for concise explanations).

*Problem* 1. A function $f\colon \mathbb{R} \to \mathbb{C}$ is **1-periodic** when for all $x \in \mathbb{R}$, $f(x+1) = f(x)$. Given a real number $\kappa$, define

$$\begin{aligned} e_\kappa \colon \mathbb{R} &\longrightarrow \mathbb{C} \\ x &\longmapsto e^{2\pi i \kappa x}. \end{aligned}$$

For which values of $\kappa$ is $e_\kappa$ 1-periodic, and for which is it not? Prove your assertion.

*Problem* 2. For this question only, do *not* worry about whether infinite sums converge.

(a) Given $c\colon \mathbb{Z} \to \mathbb{C}$ and $c_k := c(k)$, show that

$$\sum_{k\in\mathbb{Z}} c_k e_k(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(2\pi n x) + \sum_{n\geq 1} b_n \sin(2\pi n x)$$

for some $a\colon \mathbb{N} \to \mathbb{C}$ and $b\colon \mathbb{Z}_{\geq 1} \to \mathbb{C}$. Your answer should include an expression for $c_k$ in terms of $a_n$'s and $b_n$'s.

(b) Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is 1-periodic and integrable over $[0,1]$. For $k \in \mathbb{Z}$, set

$$c_k := \int_0^1 f(x) e_{-k}(x)\, dx$$

and for $n \in \mathbb{N}$, set

$$\begin{aligned} a_n &:= 2\int_0^1 f(x)\cos(2\pi n x)\, dx \\ b_n &:= 2\int_0^1 f(x)\sin(2\pi n x)\, dx. \end{aligned}$$

Show that $a, b, c$ satisfy the equations from part (a).

*Problem* 3. Prove that for all $m, n \in \mathbb{Z}$,

$$\int_0^1 e_m(x)\overline{e_n(x)}\, dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

What does this say about the set $\{e_n \mid n \in \mathbb{Z}\}$ in the language of inner product spaces?

*Problem* 4. Show that there is no continuous function $\delta\colon [-1/2, 1/2] \to \mathbb{R}$ with the following property: for all continuous functions $f\colon [-1/2, 1/2] \to \mathbb{R}$,

$$\int_{-1/2}^{1/2} f(x)\delta(x)\, dx = f(0).$$

*Instructor's note*: This shows that a continuous 'delta function' does not exist, and indicates we may need to broaden our notion of function in order to access such behavior. This leads to the notion of a *distribution*.

*Problem* 5. Show that the standard sesquilinear form $\langle\ ,\ \rangle$ is *not* positive definite on the $\mathbb{C}$-vector space of Riemann integrable functions $[0,1] \to \mathbb{C}$.