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Poisson summation as a trace

For $P \subseteq \mathbb{R}^d$ compact with positive volume and

$$K : P \times P \longrightarrow \mathbb{C}$$

continuous, define

$$T_K : L^2(P) \longrightarrow L^2(P)$$

$$f \longmapsto \left(x \mapsto \int_P K(x,y) f(y)\, dy \right).$$

$$= \int_P \left(\int_P K(x,y) f(y)\, dy \right) \overline{f(x)}\, dx$$

$$= \int_{P\times P} K(x,y) f(y) \overline{f(x)}\, dx\, dy$$

Call K a kernel.

$$\langle T_K f, f \rangle = \int_P T_K(f)(x) \cdot \overline{f(x)}\, dx$$

Check: T_K is linear

$$\langle f, T_K f \rangle = \int_P f(x) \cdot \overline{T_K f(x)}\, dx$$

Recall T_K is positive when $\langle T_K f, f\rangle > 0$ for $f \neq 0$.

Check T_K is self-adjoint (aka Hermitian) when $K(x,y) = \overline{K(y,x)}$.

$$\langle T_K f, f\rangle = \langle f, T_K f\rangle$$

By the spectral theorem, T_K has an orthonormal basis of eigenvectors $\{v_1, v_2, \dots\}$ with $T_K v_i = \lambda_i v_i$.

Thm [Mercer 1909] Suppose T_K is a positive self-adjoint operator on $P \subseteq \mathbb{R}^d$ compact. Then

$$K(x,y) = \sum_{n \geq 1} \lambda_n v_n(x) \overline{v_n(y)}$$

with abolute uniform conv. $\square$

Defn The trace of T_K is $\mathrm{Tr}(T_K) := \sum_{n=1}^{\infty} \lambda_n$.

$\left(\mathrm{Tr}(T) = \sum_{i \in \mathbb{N}} \langle T e_i, e_i \rangle \right.$

$\left. (e_i) \text{ o.n. basis of } \mathcal{H} \right)$

For T_K as in Mercer's thm,

$$\int_P K(x,x)\, dx = \int_P \sum_{n=1}^{\infty} \lambda_n v_n(x) \overline{v_n(x)} = \sum_{k=1}^{\infty} \lambda_n \int_P |v_n(x)|^2 dx = \sum_{k=1}^{\infty} \lambda_n .$$

So in this case $\mathrm{Tr}(T_K) = \int_P K(x,x)\, dx$ as well.

$\circ \circ \{ \mathbb{R}^d / \mathbb{Z}^d = \mathbb{T}^d$

Set $P = [0,1]^d$, $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ Schwartz.

Consider the linear operator

$$L_f : L^2([0,1]^d) \longrightarrow L^2(\mathbb{R}^d)$$

$$g \longmapsto \left(f*g : x \longmapsto \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \right).$$

periodization of g.

Compute

$$L_f(g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{[0,1]^d - n} f(x-y)\, g(y)\, dy$$

$= g(y-n)$

$$= \sum_{n \in \mathbb{Z}^d} \int_{[0,1]^d} f(x-y+n)\, g(y)\, dy$$

$$= \int_{[0,1]^d} \left(\sum_{n \in \mathbb{Z}^d} f(x-y+n) \right) g(y)\, dy$$

Set $K(x,y) = \sum_{n \in \mathbb{Z}^d} f(x-y+n)$. Then $L_f = T_K$.

<u>Claim</u> Eigenfunctions of L_f are $e_k : x \mapsto e^{2\pi i x \cdot k}$ for $k \in \mathbb{Z}^d$.

<u>Check</u> $L_f(e_k)(x) = \int_{\mathbb{R}^d} f(y)\, e^{2\pi i (x-y)\cdot k}\, dy$ $\qquad$ $(f * g = g * f)$

$$= e^{2\pi i x \cdot k} \int_{\mathbb{R}^d} f(y)\, e^{-2\pi i y \cdot k}\, dy$$

$$= \hat{f}(k)\, e_k(x) \implies \hat{f}(k),\ k \in \mathbb{Z}^d \text{ are eigenvalues.}$$

Since the e_k are a basis for $L^2(\mathbb{T}^d)$, $\{e_k \mid k \in \mathbb{Z}^d\}$ is an eigenbasis for L_f with associated eigenvalues $\{\hat{f}(k) \mid k \in \mathbb{Z}^d\}$.

$$\langle T v_i, v_i \rangle = \lambda_i \langle v_i, v_i \rangle = \lambda_i$$

It follows that $\operatorname{Tr}(L_f) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k)$.

Now suppose L_f is self-adjoint and positive.

$\sum_n f(x-x+n) = \sum f(n)$

Then $\displaystyle \mathrm{Tr}(L_f) = \int_{[0,1]^d} K(x,x)\, dx = \int_{[0,1]^d} \sum_{n\in\mathbb{Z}^d} f(n)\, dx$

$$= \sum_{n\in\mathbb{Z}^d} f(n) \int_{[0,1]^d} dx$$

$$= \sum_{n\in\mathbb{Z}^d} f(n).$$

So in this case, $\displaystyle \sum_{n\in\mathbb{Z}^d} f(n) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)$!

<u>Note</u> For $\mathbb{Z}^d$ replaced with $\mathcal{L} \subseteq \mathbb{R}^d$ full rank lattice contributes $\det \mathcal{L}$

Q For which f is L_f positive?

Asking for $\langle f*g, g\rangle > 0$ for $g \neq 0$.

$$\langle f*g, g\rangle = \int_{\mathbb{R}^d} (f*g)(x)\,\overline{g(x)}\,dx$$

$$= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy\right)\overline{g(x)}\,dx$$

This perspective for $\mathbb{Z}^d \leq \mathbb{R}^d$ generalizes to "cofinite discrete subgroup" $\Gamma \leq G \rightsquigarrow$ Selberg trace formula.