2025. IV. 14

Extremal bodies

Recall: A lattice $\mathcal{L} \subseteq \mathbb{R}^d$ is the $\mathbb{Z}$-linear span of linearly independent vectors $\{v_1, \dots, v_m\} \subseteq \mathbb{R}^d$:

$$\mathcal{L} = \{ n_1 v_1 + \cdots + n_m v_m \mid n_j \in \mathbb{Z} \}$$

$$= M \cdot \mathbb{Z}^m \quad \text{for} \quad M = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix} \in \mathbb{R}^{d \times m}$$

Call $\{v_1, \dots, v_m\}$ a basis for $\mathcal{L}$,

m = rank of $\mathcal{L}$

$\det \mathcal{L} = |\det M|$ for $\mathcal{L}$ full rank

Fact $\det \mathcal{L}$ is independent of basis choice.

Defn The **dual lattice** is

$$\mathcal{L}^* = \{ x \in \mathbb{R}^d \mid x \cdot n \in \mathbb{Z} \ \forall n \in \mathcal{L} \}$$

$$= M^{-T} \cdot \mathbb{Z}^d .$$

Thm [Poisson summation for general full rank lattices]

Suppose $\mathcal{L} \subseteq \mathbb{R}^d$ is a full rank lattice and $f : \mathbb{R}^d \to \mathbb{C}$ satisfies Poisson summation (for $\mathbb{Z}^d$). Then $\forall x \in \mathbb{R}^d$,

$$\sum_{n \in \mathcal{L}} f(n+x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{f}(m) e^{2\pi i x \cdot m}$$

and

$$\sum_{n \in \mathcal{L}} f(n) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{f}(\xi) .$$

<u>Pf</u> Take $M \in GL_d(\mathbb{R})$ with $\mathcal{L} = M \cdot \mathbb{Z}^d$, $\det \mathcal{L} = |\det M|$, $\mathcal{L}^* = M^{-T} \cdot \mathbb{Z}^d$.

By Poisson summation for $\mathbb{Z}^d$,

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi)$$

Now

$$\sum_{n \in \mathcal{L}} f(n) = \sum_{k \in \mathbb{Z}^d} f(M \cdot k)$$

$$= \sum_{k \in \mathbb{Z}^d} (f \circ M)(k)$$

$$= \sum_{\xi \in \mathbb{Z}^d} \widehat{(f \circ M)}(\xi) \qquad [\text{Poisson for } f \circ M]$$

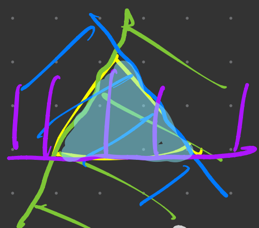
$$= \sum_{\xi \in \mathbb{Z}^d} \frac{1}{|\det M|} \hat{f}(M^{-T}\xi)$$

Defn $P = \bigcap \underbrace{\text{halfspaces}}$

$N \in \mathbb{R}^d$

$v + \{x \in \mathbb{R} \mid N \cdot x \geq 0\}$

$$= \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{f}(\xi). \quad \square$$



Defn A polytope $P \in \mathbb{R}^d$ k-tiles $\mathbb{R}^d$ using translations $\mathcal{L}$ when for some $k \in \mathbb{Z}_{>0}$,

$$\sum_{n \in \mathcal{L}} 1_{P+n}(x) = k \qquad \forall x \in \mathbb{R}^d \setminus (\partial P + \mathcal{L})$$

Note For $k = 1$, this is the standard notion of $\mathcal{L}$-periodic tiling.

Thm Suppose $P \subseteq \mathbb{R}^d$ is compact with $\operatorname{vol} P > 0$. TFAE:

(a) P k-tiles $\mathbb{R}^d$ via $\mathcal{L}$

(b) $\hat{1}_P(\xi) = 0 \quad \forall \xi \in \mathcal{L}^* \setminus 0$.

Either condition implies $k = \dfrac{\operatorname{vol} P}{\det \mathcal{L}}$.

Pf We have P k-tiles via $\mathcal{L}$ iff $\sum_{n \in \mathcal{L}} 1_{P+n}(x) = k$ for $x \in \mathbb{R}^d \setminus (\partial P + \mathcal{L})$

Observe $1_{P+n}(x) = 1 \iff 1_P(x-n) = 1$, so this is equivalent to

$$\sum_{n \in \mathcal{L}} 1_P(x-n) = k \quad \text{for } x \in \mathbb{R}^d \setminus (\partial P + \mathcal{L}).$$

Set $F(x) := \sum_{n \in \mathcal{L}} 1_P(x-n)$ which is $\mathcal{L}$-periodic.

$\sum_{n \in \mathcal{L}} f(n+x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{f}(m) e^{2\pi i x \cdot m}$

By Poisson summation*,

$$F(x) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{1}_P(\xi) e^{2\pi i \xi \cdot x}$$

To make rigorous, convolve with Dirichlet kernels...

Observe $\hat{1}_P(0) = \operatorname{vol} P$ so

$$F(x) = \frac{\operatorname{vol} P}{\det \mathcal{L}} + \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* \setminus 0} \hat{1}_P(\xi) e^{2\pi i \xi \cdot x}$$

Thus $\hat{1}_P(\xi) = 0$ for all $\xi \in \mathcal{L}^* \setminus 0$ iff $F(x) = \frac{\operatorname{vol} P}{\det \mathcal{L}} = k \in \mathbb{Z}_{>0}$.

compact subset

Defn An *extremal body* relative to a lattice $\mathcal{L}$ is a convex symmetric body K containing exactly one lattice point of $\mathcal{L}$ in its interior and such that

$$\operatorname{vol} K = 2^d (\det \mathcal{L}).$$

Thm Let K be convex centrally symmetric subset of $\mathbb{R}^d$, $\mathcal{L} \subseteq \mathbb{R}^d$ a full rank lattice. Suppose $K^\circ \cap \mathcal{L} = 0$. Then

$$2^d \det \mathcal{L} = \operatorname{vol} K \iff \tfrac{1}{2} K \text{ tiles } \mathbb{R}^d \text{ via } \mathcal{L}.$$

Pf By Siegel's formula,

$$2^d \det \mathcal{L} = \operatorname{vol} K + \frac{4^d}{\operatorname{vol} K} \sum_{\xi \in \mathcal{L}^* \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2 .$$

Hence $2^d \det \mathcal{L} = \operatorname{vol} K \iff \hat{1}_{\frac{1}{2}K}(\xi) = 0 \quad \forall \xi \in \mathcal{L}^* \setminus 0$.

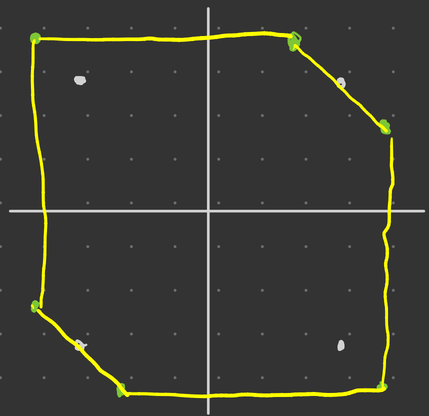
$\iff \frac{1}{2}K$ k-tiles for $k = \dfrac{\operatorname{vol}(\frac{1}{2}K)}{\det \mathcal{L}}$

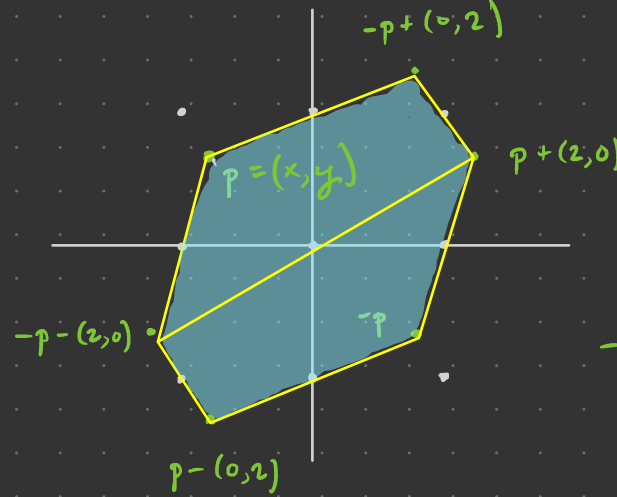
$$= \frac{\operatorname{vol} K}{2^d \det \mathcal{L}}$$

$$= 1$$

$\square$

Where is area = 4?





Shoelace

p	x	y
$-p+(0,2)$	$-x$	$-y+2$
$p+(2,0)$	$x+2$	y
$-p$	$-x$	$-y$
$p-(0,2)$	x	$y-2$
$-p-(2,0)$	$-x-2$	$-y$
p	x	y

$$x(-y+2)+xy$$
$$+\ -xy-(x+2)(y+2)$$
$$+\ \ldots$$