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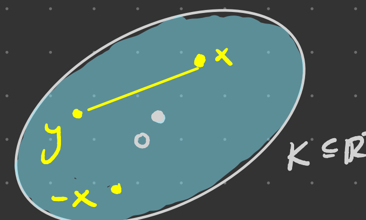
Minkowski's convex body theorem

Throughout, $K \subseteq \mathbb{R}^d$ is compact.

Call K **centrally symmetric** when

$$x \in K \iff -x \in K$$



K **convex** when $\forall x, y \in K$,

$$tx + (1-t)y \in K \quad \forall\, t \in [0,1].$$

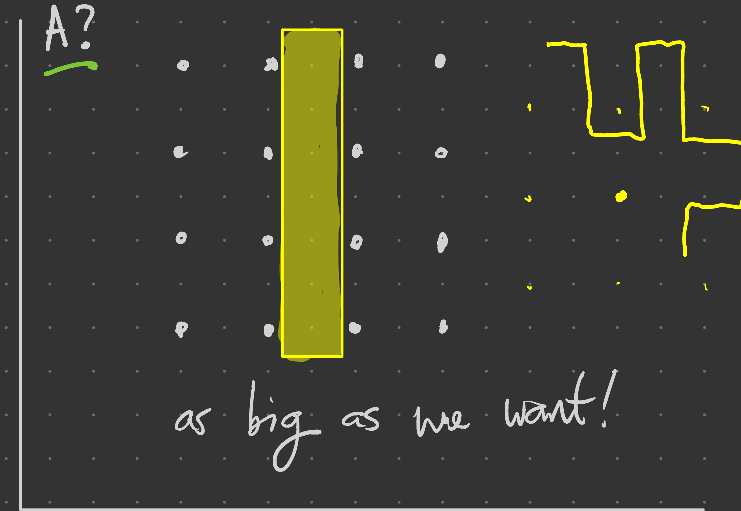
Say K **contains a lattice point** when $K \cap \mathbb{Z}^d \neq \emptyset$.

Motivating question How large must K be in order to contain a lattice point?

Thm [Minkowski] Let $K \subseteq \mathbb{R}^d$ be compact, convex, and centrally symmetric. *If* vol $K > 2^d$, then

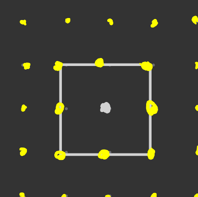
$$K^\circ \cap (\mathbb{Z}^d \setminus 0) \neq \emptyset$$

(i.e. the interior of K contains a nonzero lattice point).



Note The bound is sharp: $K = [-1,1]^d$ has volume 2^d and

$$(-1,1)^d \cap \mathbb{Z}^2 = \{(0,0)\}.$$



Note Contrapositive: $K \subseteq \mathbb{R}^d$ compact convex centrally symmetric.

If $K \cap \mathbb{Z}^d = \{0\}$, then $\operatorname{vol} K \leq 2^d$.

Minkowski's theorem follows from the following:

Thm [Siegel] $K \subseteq \mathbb{R}^d$ compact convex centrally symmetric.

If $K \cap \mathbb{Z}^d = \{0\}$, then

$$2^d = \operatorname{vol} K + \frac{4^d}{\operatorname{vol} K} \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2$$

$1_{\frac{1}{2}K}$ = char fn of $\frac{1}{2}K$

Here $rK := \{ rx \mid x \in K \}$ for $r \in \mathbb{R}$.

There's even a more general version:

<u>Thm</u> [Siegel] Let $K \subseteq \mathbb{R}^d$ be compact and assume $1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K}$ satisfies Poisson summation. If $(\frac{1}{2}K - \frac{1}{2}K)^\circ \cap \mathbb{Z}^d = \{0\}$, then

$$2^d = \operatorname{vol} K + \frac{4^d}{\operatorname{vol} K} \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2 .$$

Here $K + L = \{x+y \mid x \in K, y \in L\}$ and $\frac{1}{2}K - \frac{1}{2}K$ is the symmetrization of K.

<u>Exercise</u> Show $K-K$ is centrally symmetric.

Take $z \in K-K$. Then $z = x-y$, $x, y \in K$. Then

$-z = y - x$ and $y, x \in K$ so $-z \in K-K$.

<u>Fact</u> If K is cent sym, then $\frac{1}{2}K - \frac{1}{2}K = K$.

Pf of Thm Set $f = 1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K} \in C(\mathbb{R}^d)$. By Poisson summation,

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi).$$

By definition of f,

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y)\, 1_{-\frac{1}{2}K}(n-y)\, dy$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} 1_{\frac{1}{2}K^\circ}(y)\, 1_{-\frac{1}{2}K^\circ}(n-y)\, dy.$$

We have $y \in \frac{1}{2}K$ and $n-y \in -\frac{1}{2}K$ iff $n \in \frac{1}{2}K - \frac{1}{2}K$.

The only interior lattice point of $\frac{1}{2}K - \frac{1}{2}K$ is 0 by hypothesis. Thus only $n=0$ contributes:

$$\sum_{n \in \mathbb{Z}^d} f(n) = f(0) = \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y)\, 1_{-\frac{1}{2}K}(-y)\, dy$$

$$= \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y)\, dy$$

$$= \frac{\operatorname{vol} K}{2^d}.$$

Meanwhile,

$$\sum_{\xi\in\mathbb{Z}^d}\hat{f}(\xi) = \sum_{\xi\in\mathbb{Z}^d}\hat{1}_{\frac{1}{2}K}(\xi)\,\hat{1}_{-\frac{1}{2}K}(\xi)$$

$$f = 1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K}$$

$$\Rightarrow \hat{f}(\xi) = \hat{1}_{\frac{1}{2}K}(\xi)\,\hat{1}_{-\frac{1}{2}K}(\xi)$$

$$= \sum_{\xi\in\mathbb{Z}^d}\int_{\frac{1}{2}K} e^{2\pi i\xi\cdot x}\,dx \int_{-\frac{1}{2}K} e^{2\pi i\xi\cdot x}\,dx$$

$$= \sum_{\xi\in\mathbb{Z}^d}\int_{\frac{1}{2}K} e^{2\pi i\xi\cdot x}\,dx \int_{\frac{1}{2}K} e^{2\pi i\xi\cdot(-x)}\,dx$$

$$= \sum_{\xi\in\mathbb{Z}^d}\int_{\frac{1}{2}K} e^{2\pi i\xi\cdot x}\,dx\ \overline{\int_{\frac{1}{2}K} e^{2\pi i\xi\cdot x}\,dx}$$

$$= \left|\hat{1}_{\frac{1}{2}K}(0)\right|^2 + \sum_{\xi\in\mathbb{Z}^d\setminus 0}\left|\hat{1}_{\frac{1}{2}K}(\xi)\right|^2$$

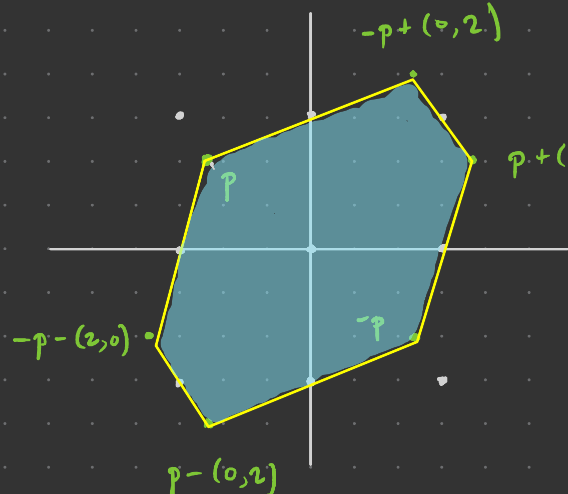
$$= \left(\frac{\operatorname{vol} K}{2^d} \right)^2 + \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2 .$$

Hence $$\frac{\operatorname{vol} K}{2^d} = \left(\frac{\operatorname{vol} K}{2^d} \right)^2 + \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2$$

$$\Rightarrow \quad 2^d = \operatorname{vol} K + \frac{2^{2d}}{\operatorname{vol} K} \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2 . \qquad \square$$

Fact For $K \subseteq \mathbb{R}^d$ compact convex, $1_K * 1_{-K}$ is "nice" (satisfies Poisson summation).

E.g. For which $p \in \mathbb{R}^2$ does the following body have no nonzero integer points in its interior?



Where is area = 4?

There is also a version of Minkowski-Siegel for general full rank lattices in $\mathbb{R}^d$: $\mathcal{L} \subseteq \mathbb{R}^d$, $\mathcal{L} \cong \mathbb{Z}^d$ as Abelian groups

Thm Suppose $K \subseteq \mathbb{R}^d$ compact with $1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K}$ "nice".

Let $\mathcal{L} \subseteq \mathbb{R}^d$ be a full rank lattice with dual lattice

$$\mathcal{L}^* = \{x \in \mathbb{R}^d \mid x \cdot n \in \mathbb{Z} \text{ for all } n \in \mathcal{L}\}.$$
$$= M^{-T}\mathcal{L}.$$

$(\mathbb{Z}^d)^* = \mathbb{Z}^d$

If $(\frac{1}{2}K - \frac{1}{2}K)^\circ \cap \mathcal{L} = \{0\}$, then

$$2^d \det \mathcal{L} = \operatorname{vol} K + \frac{4^d}{\operatorname{vol} K} \sum_{\xi \in \mathcal{L}^* \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2.$$

$\det M$, M lin trans $\mathbb{R}^d \to \mathbb{R}^d$ s.t. $M(\mathbb{Z}^d) = \mathcal{L}$.

Note

$f \in L^1(\mathbb{R}^d)$

$M : \mathbb{R}^d \to \mathbb{R}^d$ linear

$\widehat{f \circ M}$ involves M^{-T}