2025. IV. 9

Fourier analysis on $\mathbb{R}^d$

Recall $f \in L^1(A)$,

$$\begin{aligned} \hat{f} : \hat{A} &\longrightarrow \mathbb{C} \\ \chi &\longmapsto \int_A f(x)\overline{\chi(x)}\, dx . \end{aligned}$$

For $A = \mathbb{R}^d$, $\hat{A} \cong \mathbb{R}^d$ via

$$\xi \in \mathbb{R}^d \longmapsto \left(\begin{aligned} \mathbb{R}^d &\longrightarrow S^1 \\ x &\longmapsto e^{2\pi i \underbrace{\xi \cdot x}_{\text{standard dot product}}} \end{aligned} \right)$$

We may thus rewrite

$$\begin{aligned} \hat{f} : \mathbb{R}^d &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x}\, dx \end{aligned}$$

Lemma If $f \in L^1(\mathbb{R}^d)$, then (a) $\|\hat{f}\|_\infty \leq \|f\|_1$, and

(b) $\hat{f}$ is uniformly continuous.

Pf For (a), given $\xi \in \mathbb{R}^d$, we have

$$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} \, dx \right|$$

$$\leq \int_{\mathbb{R}^d} \left| f(x) e^{-2\pi i \xi \cdot x} \right| dx$$

$$= \int_{\mathbb{R}^d} |f(x)| \, dx$$

$$= \|f\|_1 .$$

For (b), fix $\xi, h \in \mathbb{R}^d$. Then

$$\left|\hat{f}(\xi+h) - \hat{f}(\xi)\right| = \left|\int_{\mathbb{R}^d} f(x)\left(e^{-2\pi i(\xi+h)\cdot x} - e^{-2\pi i \xi\cdot x}\right) dx\right|$$

$$= \left|\int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi\cdot x}\left(e^{-2\pi i h\cdot x} - 1\right) dx\right|$$

$$\leq \int_{\mathbb{R}^d} |f(x)| \cdot \left|e^{-2\pi i h\cdot x} - 1\right| dx.$$

Let $g_h(x) := f(x)\left(e^{-2\pi i h\cdot x} - 1\right)$. Then $|g_h(x)| \leq 2|f(x)|$

and $\lim_{h\to 0} g_h(x) = 0$. Since g_h dominated by L^1 function $2f$, we can use dominated convergence to conclude

$$\lim_{h\to 0} \int_{\mathbb{R}^d} |g_h(x)|\, dx = \int_{\mathbb{R}^d} \left(\lim_{h\to 0} |g_h(x)|\right) dx = 0$$

Thus $|\hat{f}(\xi+h) - \hat{f}(\xi)| \xrightarrow[h\to 0]{} 0$ uniformly in ξ. $\square$

E.g. Take $S \subseteq \mathbb{R}^d$ bounded and measurable. Then $1_S \in L^1(\mathbb{R}^d)$

characteristic function of S

and $|\hat{1}_S(\xi)| \leq \|1_S\|_1 = \int_S dx = \mu(S)$.

$\hat{1}_S \notin L^1(\mathbb{R}^d)$ in general.

Pf Suppose for contradiction $\hat{1}_S \in L^1(\mathbb{R}^d)$. Then

$$\hat{\hat{1}}_S(x) = 1_S(-x)$$

is continuous, but of course 1_S is not! $\square$

E.g. If $S = \prod_{i=1}^{d} S_i$, $S_i \subseteq \mathbb{R}$, then $\hat{1}_S(\xi) = \prod_{i=1}^{d} \hat{1}_{S_i}(\xi_i)$ by

$$\hat{1}_S = \int_S e^{-2\pi i \xi \cdot x}\, dx = \int_S e^{-2\pi i (\xi_1 x_1 + \cdots + \xi_d x_d)}\, dx = \int_S \prod_k e^{-2\pi i \xi_k x_k}\, dx$$

Recall

$$\hat{1}_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) = \frac{\sin(\pi \xi)}{\pi \xi} = \mathrm{sinc}(\xi)$$

so

$$\hat{1}_{[-1/2, 1/2]^d}(\xi) = \frac{1}{\pi^d} \prod \frac{\sin(\pi \xi_i)}{\xi_i}.$$

Parlor trick

Since $\hat{\hat{f}}(x) = f(-x)$, we have $\hat{\hat{f}}(0) = f(0)$

i.e. $$f(0) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, \underbrace{e^{-2\pi i \xi \cdot 0}}_{1}\, d\xi$$

$$= \int_{\mathbb{R}^d} \hat{f}(\xi)\, d\xi.$$

For $f = 1_{[-1/2, 1/2]^d}$, this says

$$1 = \int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{\sin(\pi \xi_i)}{\pi \xi_i}\, d\xi$$

Multivariable Poisson summation

For "nice" $f \in L^1(\mathbb{R}^d)$,

$$\sum_{n \in \mathbb{Z}^d} f(n+x) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \qquad \text{for all } x \in \mathbb{R}^d.$$

In particular, $(x=0)$

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi).$$

Idea: Look at $f = 1_S$ for $S \subseteq \mathbb{R}^d$.

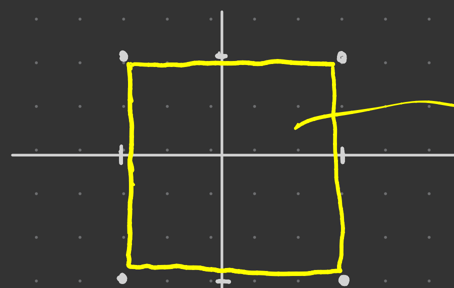
Then $\displaystyle\sum_{n \in \mathbb{Z}^d} 1_S(n) = \#\{n \in \mathbb{Z}^d \mid n \in S\}$.

Challenge: count on the Fourier side as well!

Minkowski's Thm (baby version)

Suppose $C \subseteq \mathbb{R}^d$ is a compact convex! subset that is centrally symmetric ($x \in C \Rightarrow -x \in C$). If 0 is the only lattice point ($\mathbb{Z}^d$) in $C \setminus \partial C$, then

$$\mathrm{vol}(C) \leq 2^d.$$



$\mathrm{vol} \leq 4 = 2^2$

Siegel: $\underbrace{2^d - \mathrm{vol}(C) = ?}$

Siegel tells us what this is!

$$\frac{1}{2}K - \frac{1}{2}K = \left\{ \frac{1}{2}(x-y) \;\middle|\; x,y \in K \right\}$$

symmetrization