2025. IV. 7

Convolution & Plancherel's Theorem for LCA groups

A LCA gp.

Defineorem For $f, g \in L^1_{bc}(A)$, the convolution

$$\begin{aligned} f * g : A &\longrightarrow \mathbb{C} \\ x &\longmapsto \int_A f(xy^{-1})\, g(y)\, dy \end{aligned}$$

exists and is in $L^1_{bc}(A)$.

Pf Assume $|f(x)| \le C$ $\forall x \in A$. Then

$$\int_A |f(xy^{-1})\, g(y)|\, dy \le C \int_A |g(y)|\, dy = C\|g\|_1$$

so the integral exists and $f * g$ is bounded.

Now for continuity: let $x_0 \in A$, assume $|f(x)|, |g(x)| \le C \;\; \forall x \in A$,

assume $g \neq 0$. Given $\varepsilon > 0$, $\exists\, \varphi \in C_c^+(A)$ s.t. $\varphi \le |g|$

and $\int_A (|g(y)| - \varphi(y))\, dy < \frac{\varepsilon}{4C}$ by density of $C_c(A)$ in $L^1_{loc}(A)$

K compact

Since f cts, it is unif cts on compacts. Thus $\exists$ open nbhd V of $e \in A$

such that $xy^{-1} \in V$, $x \in x_0 (\operatorname{supp} \varphi)^{-1}$ $\Longrightarrow$

$$|f(xy^{-1}) - f(x_0 y^{-1})| < \frac{\varepsilon}{2\|g\|_1}.$$

Thus for $x \in V x_0$,

$$\int_A |f(xy^{-1}) - f(x_0 y^{-1})|\, \varphi(y)\, dy \le \frac{\varepsilon}{2\|g_1\|} \overbrace{\int_A \varphi(y)\, dy}^{\le \|g\|_1} \le \frac{\varepsilon}{2},$$

and

$$\int_A \underbrace{|f(xy^{-1}) - f(x_0 y^{-1})|}_{\leq |f(xy^{-1})| + |f(x_0 y^{-1})| \leq 2C} \left(|g(y)| - \varphi(y)\right) dy$$

$$\leq 2C \int_A \left(|g(y)| - \varphi(y)\right) dy < \frac{\varepsilon}{2}.$$

Thus for $x \in x_0 V$,

$$|f*g(x) - f*g(x_0)| = \left|\int_A \left(f(xy^{-1}) - f(x_0 y^{-1})\right) g(y)\, dy\right|$$

$$\leq \int_A |f(xy^{-1}) - f(x_0 y^{-1})|\, |g(y)|\, dy$$

$$= \int_A |f(xy^{-1}) - f(x_0 y^{-1})| \left(|g(y)| - \varphi(y) + \varphi(y)\right) dy$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f * g$ is continuous.

To see $f*g$ is L^1, compute

$$\|f*g\|_1 = \int_A |f*g(x)|\,dx$$

$$= \int_A \left| \int_A f(xy^{-1})\, g(y)\, dy \right| dx$$

$$\leq \int_A \int_A |f(xy^{-1})\, g(y)|\, dy\, dx$$

Fubini

$$= \int_A \int_A |f(xy^{-1})\, g(y)|\, dx\, dy$$

$$= \int_A |g(y)| \left(\int_A f(xy^{-1})\, dx \right) dy$$

$\|$ invariance

$$= \int_A |g(y)|\, dy \cdot \int_A |f(x)|\, dx$$

$$= \|g\|_1 \|f\|_1 < \infty .$$

Hence $f * g \in L^1_{bc}(A)$. $\square$

Recall The Fourier transform of $f \in L^1_{bc}(A)$ is

$$\hat{f} : \hat{A} \longrightarrow \mathbb{C}$$

$$\chi \longmapsto \int_A f(x) \overline{\chi(x)}\, dx .$$

Thm For $f, g \in L^1_{bc}(A)$, $\widehat{f * g}(\chi) = \hat{f}(\chi)\hat{g}(\chi)$.

Pf Let us compute:

$$\widehat{f*g}(\chi) = \int_A f*g(x)\,\overline{\chi(x)}\,dx$$

$$= \int_A \left(\int_A f(xy^{-1})\,g(y)\,dy\right)\overline{\chi(x)}\,dx$$

Fubini

$$= \int_A \int_A f(xy^{-1})\,g(y)\,\overline{\chi(x)}\,dx\,dy$$

$x \to xy$ [invariant]

$$= \int_A \int_A f(x)\,g(y)\,\underbrace{\overline{\chi(yx)}}_{\overline{\chi(y)}\,\overline{\chi(x)}}\,dx\,dy$$

$$= \int_A f(x)\,\overline{\chi(x)}\,dx \cdot \int_A g(y)\,\overline{\chi(y)}\,dy$$

$$= \hat{f}(x)\,\hat{g}(x). \qquad \square$$

Plancherel's Theorem for LCA groups Let A be an LCA group.

For $f \in L^1_{\mathrm{loc}}(A)$, $\hat{f} \in L^2_{\mathrm{loc}}(\hat{A})$. There is a unique Haar measure on $\hat{A}$ such that

$$\|f\|_2 = \|\hat{f}\|_2.$$

Thus the Fourier transform extends to a Hilbert space isomorphism

$$L^2(A) \cong L^2(\hat{A}).$$

We will prove this for A discrete (which in turn proves the A compact case since $\hat{\hat{(\,)}} \cong \mathrm{id}$).

Lemma Let A be a compact Abelian group. Fix a Haar integral such that $\int_A 1\,dx = 1$. Then $\forall \chi, \eta \in \hat{A}$,

$$\int_A \chi(x)\overline{\eta(x)}\,dx = \begin{cases} 1 & \text{if } \chi = \eta \\ 0 & \text{o/w.} \end{cases}$$

Pf If $\chi = \eta$, then $\int_A \chi(x)\overline{\chi(x)}\,dx = \int_A |\chi(x)|^2\,dx = \int_A dx = 1$.

Suppose $\chi \neq \eta$. Then $\alpha := \chi\bar{\eta} = \chi\eta^{-1} \neq 1$. Take $a \in A$ with $\alpha(a) \neq 1$.

Then $\alpha(a)\int_A \alpha(x)\,dx = \int_A \alpha(ax)\,dx = \int_A \alpha(x)\,dx$

invariance

so $\quad (\alpha(a)-1)\int_A \alpha(x)\,dx = 0 \implies \int_A \alpha(x)\,dx = 0.$ $\square$

<u>Lemma</u> Suppose A is discrete. For every $g \in L^1_{bc}(A)$,

$\hat{g} \in L^1_{bc}(\hat{A}) = C(\hat{A})$, and for every $a \in A$,

$$\hat{\hat{g}}(\mathrm{eval}_a) = g(a^{-1}).$$

$$\begin{tikzcd} A \arrow[r, "\cong"] & \hat{\hat{A}} \\ a \arrow[r, mapsto] & \mathrm{eval}_a \end{tikzcd} \qquad \begin{tikzcd} \hat{A} \arrow[d] & x \arrow[d, mapsto] \\ \mathbb{C} & x(a) \end{tikzcd}$$

<u>Pf</u> For Haar integral on A, have $\int_A f(x)\,dx = \sum_{a\in A} f(a)$.

compact!

For Haar integral on $\hat{A}$ normalize with $\int_{\hat{A}} 1\,dx = 1$.

Compute $\hat{\hat{g}}(\mathrm{eval}_a) = \int_{\hat{A}} \hat{g}(x)\,\overline{\mathrm{eval}_a(x)}\,dx$

$$= \int_{\hat{A}} \left(\sum_{b \in A} g(b) \overline{\chi(b)} \right) \overline{\mathrm{eval}_a(\chi)} \, d\chi$$

$$= \int_{\hat{A}} \sum_{b \in A} g(b) \overline{\chi(b)} \, \overline{\chi(a)} \, d\chi$$

$$\overline{\chi(b)} = \chi(b)^{-1}$$

$$= \chi(b^{-1})$$

$$b \to b^{-1}$$

$$= \int_{\hat{A}} \sum_{b \in A} g(b^{-1}) \chi(b) \overline{\chi(a)} \, d\chi$$

Fubini

$$= \sum_{b \in A} g(b^{-1}) \underbrace{\int_{\hat{A}} \mathrm{eval}_b(\chi) \overline{\mathrm{eval}_a}(\chi) \, d\chi}_{\begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}}$$

$$= g(a^{-1}). \quad \square$$

Pf of Plancherel for A discrete

Let $f \in L^1_{bc}(A)$, set $\tilde{f}(x) = \overline{f(x^{-1})}$.

Set $g = \tilde{f} * f$. Then $g(x) = \int_A \overline{f(yx^{-1})}\, f(y)\, dy$, so

$g(e) = \|f\|_2^2$. We have $\hat{g}(\chi) = \underbrace{\hat{\tilde{f}}(\chi)\, \hat{f}(\chi) = \overline{\hat{f}(\chi)}}_{\text{exc}}\, \hat{f}(\chi) = |\hat{f}(\chi)|^2$.

Thus
$$
\begin{aligned}
\|f\|_2^2 &= g(e) = \hat{\hat{g}}(\mathrm{eval}_e) \\
&= \int_{\hat{A}} \hat{g}(\chi)\, \underbrace{\overline{\chi(e)}}_{1}\, d\chi \\
&= \int_{\hat{A}} |\hat{f}(\chi)|^2\, d\chi \\
&= \|\hat{f}\|_2^2 . \quad \square
\end{aligned}
$$