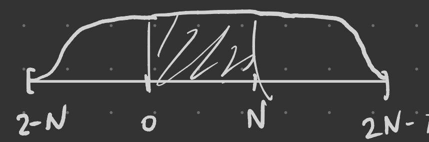
2025. IV. 4

Guest lecture: Josh Zahl

Application to number theory:

$x_1 = y_1 + y_2 - x_2$



$$x_1 + x_2 = y_1 + y_2 \qquad 1 \le x_i, y_i \le N$$

Number of solutions $J_2(N)$ grows like N^3

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$$

more interesting

$$2x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$

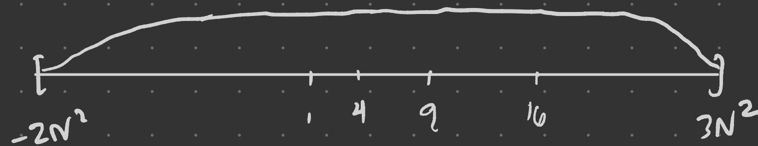
$\sim N^4$ solns

$$N^3 \le J_3(N) \le N^4$$

$x_i = y_i$

solve for x_1, x_2 in terms of other variables

Heuristic: $J_3(N) \sim N^3$



$$x_1 + x_2 + \cdots + x_s = y_1 + y_2 + \cdots + y_s$$
$$x_1^2 + x_2^2 + \cdots + x_s^2 = y_1^2 + y_2^2 + \cdots + y_s^2$$
$$\vdots$$
$$x_1^k + x_2^k + \cdots + x_s^k = y_1^k + y_2^k + \cdots + y_s^k$$

$1 \le x_i, y_i \le N$ $\qquad$ # sol'ns $=: J_{s,k}(N)$

$s! \cdot N^s \le J_{s,k}(N)$ in general

If $s = \frac{1}{2}k(k+1)$, then $J_{s,k}(N) \sim N^s$ — 2016 proof via Fourier analysis

Notation $\quad e(\xi) = e^{2\pi i \xi}$

$$n \in \mathbb{Z}, \quad \int_0^1 e(n\xi)\, d\xi = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Subtracting RHS from LHS gives expressions that are integers which are 0 only when system solved.

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$$

$$\int_0^1 \left| \sum_{x=1}^{N} e(x\xi) \right|^2 d\xi = \int_0^1 \left(\sum_{x=1}^{N} e(x\xi) \right) \overline{\left(\sum_{y=1}^{N} e(y\xi) \right)}\, d\xi$$

$$= \int_0^1 \left(\sum_{x=1}^{N} e(x\xi) \right) \left(\sum_{y=1}^{N} e(-y\xi) \right) d\xi$$

$$= \int_0^1 \sum_{x=1}^{N} \sum_{y=1}^{N} e(x\xi)\, e(-y\xi)\, d\xi$$

$$= \int_0^1 \sum_{x=1}^{N} \sum_{y=1}^{N} e((x-y)\xi)\, d\xi$$

$$= \sum_{x=1}^{N} \sum_{y=1}^{N} \int_0^1 e((x-y)\xi)\, d\xi$$

$= N$ counts when $x=y$.

$$\int_{[0,1]^2} \left| \sum_{x=1}^{N} e(x\xi_1 + x^2\xi_2) \right|^{\overset{=3\cdot 2}{6}} d\xi$$

$$= \int_{[0,1]^2} \sum_{\substack{x_1,x_2,x_3=1 \\ y_1,y_2,y_3=1}}^{N} e\left(x_1\xi_1 + x_1^2\xi_2 + x_2\xi_1 + x_2^2\xi_2 + x_3\xi_1 + x_3^2\xi_2 - y_1\xi_1 - y_1^2\xi_2 - y_2\xi_1 - y_2^2\xi_2 - y_3\xi_1 - y_3^2\xi_2\right) d\xi_1\, d\xi_2$$

$$= J_{3,2}(N)$$

$$\boxed{\int_{[0,1]^2} e(n_1\xi_1 + n_2\xi_2)\, d\xi_1\, d\xi_2 = \begin{cases} 1 & \text{if } n_1 = n_2 = 0 \\ 0 & \text{o/w.} \end{cases}}$$

More generally, $J_{s,k}(N) = \|f\|_{L^{2s}([0,1]^k)}^{2s}$

$\left(f = \sum_{x=1}^{N} e(x\xi_1 + x^2\xi_2 + \cdots)\right)$

Thm [Bourgain-Demeter-Guth] For $s = \frac{1}{2}k(k+1)$,

$$\|f\|_{L^{2s}([0,1]^k)}^{2s} \lesssim \left(\sum_{n=1}^{N} \underbrace{\|f_{\theta_n}\|_{L^{2s}([0,1]^k)}^{2}}_{1} \right)^{s} = N^s$$

for $f = \sum f_{\theta_n}$,

curve supported on moment curve $(t, t^2, \dots, t^k)$