Fubini

2025. IV. 2

G, H LC groups then $G \times H$ is an LC group

Thm Let $I_G : C_c(G) \longrightarrow \mathbb{C}$ be a Haar integral on G.

$$f \longmapsto \int_G f$$

Then $\forall f \in C_c(G \times H)$, the function

$$H \longrightarrow \mathbb{C}$$

$$y \longmapsto I_G(f(-, y)) = \int_G f(x,y)\, dx$$

lies in $C_c(H)$. Let $I_H = \int_H$ be a Haar integral on H.

Then a Haar integral on $G \times H$ is given by

$$I(f) = \int_H \int_G f(x,y)\, dx\, dy$$

iterated integral = Haar integral

This can be performed in opposite order, yielding the same result, so

$$\int_H \int_G f(x,y)\,dx\,dy = \int_G \int_H f(x,y)\,dy\,dx.$$

Proof: Endow $C_c(G)$ with topology of uniform convergence:

$f_n \xrightarrow[n\to\infty]{} f$ means $\underset{C_c(G)}{f_n}$ conv unif to $\underset{C_c(G)}{f}$ and $\exists K \subseteq G$ compact with $\mathrm{supp}(f_n) \subseteq K$ $\forall n$.

Lemma: $I_G : C_c(G) \longrightarrow \mathbb{C}$ is cts ($f_n \to f$ in $C_c(G) \Rightarrow I(f_n) \to I(f)$ in $\mathbb{C}$).

Pf Take $f_n \to f$, $K \subseteq G$ as above. Let $\chi \in C_c^+(G)$ satisfy $\chi|_K = 1$.
Define $c := \int_G \chi$. Fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ s.t. $n \geq N$
implies $|f_n(x) - f(x)| < \frac{\varepsilon}{c}$ $\forall x \in G$. Then for $n \geq N$,

$$\begin{aligned}
\left| \int_G f_n - \int_G f \right| &= \left| \int_G (f_n - f) \right| \\
&\leq \int_G |f_n - f| \\
&= \int_G \chi(x) \, |f_n(x) - f(x)| \, dx \\
&< \frac{\varepsilon}{c} \int_G \chi = \varepsilon . \quad \square
\end{aligned}$$

Lemma Fix $f \in C_c(G \times H)$. Then the function

$$\begin{aligned} H &\longrightarrow \mathbb{C} \\ y &\longmapsto \int_G f(x,y)\,dx \end{aligned}$$

is in $C_c(H)$.

Pf Suppose $y_n \to y$ in H, let $f_n(x) = f(x, y_n)$. By uniform continuity of f (why?) ✓, $\underset{f(-,y_n)}{\underset{\parallel}{f_n}} \longrightarrow f(-,y)$ in $C_c(G)$.

Thus $y \longmapsto \int_G f(-,y)$ is cts.

The projection $G \times H \xrightarrow[p_2]{} H$ is cts, so $p_2(\mathrm{supp}(f)) \subseteq H$ is compact.

Thus $y \longmapsto \int_G f(-,y)$ has compact support. $\square$

Now define $I_1(f) := \int_H \int_G f(x,y)\, dx\, dy$.

Lemma I_1 is a Haar integral on $G \times H$.

Pf For $s = (x_0, y_0) \in G \times H$,

$$I_1(L_s f) = \int_H \int_G f(x_0^{-1}x, y_0^{-1}y)\, dx\, dy$$

$$= \int_H \int_G f(x, y_0^{-1}y)\, dx\, dy \quad [\textstyle\int_G \text{ Haar}]$$

$$= \int_H \int_G f(x,y)\, dx\, dy \quad [\textstyle\int_H \text{ Haar}]$$

$$= I_1(f). \quad \square$$

Similarly, $I_2(f) := \int_G \int_H f(x,y)\,dy\,dx$ defines a Haar integral on $G \times H$. Thus $I_1(f) = c\, I_2(f)$ for some $c \geq 0$.

If $f(x,y) = g(x)h(y)$ for $g \in C_c^+(G)$, $h \in C_c^+(H)$, thin
$\quad\ \ \ast 0 \qquad\qquad\quad \ast 0$

$f \in C_c^+(G \times H)$ and $I_1(f) = \int_G g(x)\,dx \cdot \int_H h(y)\,dy = I_2(f)$;
$\ast 0$

so $c = 1$. $\square$

Let's extend to more functions:

$f : G \to \mathbb{R}_{\geq 0}$ cts. Define — no assumption on support.

$$\int_G f := \sup_{\substack{\varphi \in C_c(G) \\ 0 \leq \varphi \leq f}} \int_G \varphi \in [0, \infty].$$

Define $L^1_{bc}(G) := \left\{ f : G \to \mathbb{C} \;\middle|\; f \text{ bdd, cts, } \|f\|_1 := \int_G |f| < \infty \right\},$

$\cap$

$L^2_{bc}(G) := \left\{ f : G \to \mathbb{C} \;\middle|\; f \text{ bdd, cts, } \|f\|_2^2 := \int_G |f|^2 < \infty \right\}.$

For $f \in L^1_{bc}(G)$, write $f = u + iv$ for real-valued $u, v \in L^1_{bc}(G)$.

Then we can define

$$\int_G f = \int_G \overbrace{u_+}^{\max\{u,0\}} - \int_G \underbrace{u_-}_{\max\{-u,0\}} + i\left(\int_G v_+ - \int_G v_-\right).$$

Thm [Haar Fubini] Let $f \in L^1_{bc}(G \times H)$. Then $y \mapsto \int_G f(-,y)$ is in $L^1_{bc}(H)$ and $x \mapsto \int_H f(x,-)$ is in $L^1_{bc}(G)$.

We have
$$\int_G \int_H f(x,y)\,dy\,dx = \int_H \int_G f(x,y)\,dx\,dy. \quad \square$$

Note $L^2_{bc}(G)$ with $\langle f, g\rangle = \int_G f \cdot \bar{g}$ is an inner product space with Hilbert space completion $L^2(G)$ = Hilbert space completion of $C_c(G)$)

Fourier transforms for functions on LCA groups

A an LCA group, $\int_A$ a Haar integral

$\hat{A} = \{$ cts characters $\chi : A \to S^1 \}$.
unitary

For $f \in L^1_{loc}(A)$, define

$$\hat{f} : \hat{A} \longrightarrow \mathbb{C}$$
$$\chi \longmapsto \int_A f(x)\overline{\chi(x)}\,dx \;\text{``}{=}\text{''}\; \langle f, \chi \rangle$$

true for $f \in L^2$, A compact

Explore What does this look like for $A = S^1$, $A = \mathbb{R}$?

$$A = S^1 \Longrightarrow \hat{A} \cong \mathbb{Z}$$

$$\begin{aligned} \hat{f} : \mathbb{Z} &\longrightarrow \mathbb{C} \\ k &\longmapsto \int_{S^1} f(x)\, e^{-2\pi i k x}\, dx = \hat{f}(k) \end{aligned}$$

character

$x \longmapsto e^{2\pi i k x}$