2025. III. 31

Haar integration

Context: locally compact Hausdorff groups (LC groups)

E.g. • $G = GL_n(\mathbb{R})$ as a subspace of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$

• G an LCA group • $GL_n(\mathbb{Q}_p)$

• G any Lie group • Bohr compactification

Defn $C_c(G) = \{ f : G \to \mathbb{C} \mid f$ cts with compact support$\}$

Here $\mathrm{supp}(f) = \overline{\{x \in G \mid f(x) \neq 0\}}$

Call $f \in C_c(G)$ nonnegative $(f \geq 0)$ when $\forall x \in G$, $f(x) \in \mathbb{R}_{\geq 0}$.

A linear functional $I : C_c(G) \longrightarrow \mathbb{C}$ is an "integral"

if $f \geq 0 \Rightarrow I(f) \geq 0$.

E.g. • Riemann integral $\int_{-\infty}^{\infty} (\ )\, dx$ on $(\mathbb{R}, +)$

• $\delta_x : C_c(G) \longrightarrow \mathbb{C}$ the Dirac distribution at x.

$$f \longmapsto \delta_x(f) = f(x)$$

Fix an integral $I : C_c(G) \longrightarrow \mathbb{C}$ and write $I(f) =: \int_G f(x)\, dx$.

$$= \int_G f$$

Lemma $\left| \int_G f(x)\, dx \right| \leq \int_G |f(x)|\, dx$. ⊛

<u>Pf</u> Reduction to real-valued $f \in C_c(G)$:

First, if $f : G \to \mathbb{R} \in C_c(G)$, then $\int_G f \in \mathbb{R}$ (why?)

$f_+(x) = \max\{f(x), 0\}$ $\Rightarrow f = f_+ - f_-$
$f_-(x) = \max\{-f(x), 0\}$

Thus for gen'l f, $\mathrm{Re}\left(\int_G f\right) = \int_G \mathrm{Re}(f)$.

For $\theta \in S^1$, $\int_G \theta f = \theta \int_G f$ so multiplying f by θ doesn't change either side of ⊛. So WLOG, $\int_G f \in \mathbb{R}$.

Now suppose we have proven ⊛ for f real-valued.

Then for gen'l f, $\left|\int_G f\right| = \left|\mathrm{Re}\left(\int_G f\right)\right| = \left|\int_G \mathrm{Re}(f)\right| \underset{⊛}{\leq} \int_G |\mathrm{Re}(f)| \underset{|\mathrm{Re} f| \leq |f|}{\leq} \int_G |f|$

Now to prove $\oplus$ for real-valued f:

Let $f_\pm := \max\{\pm f, 0\}$. Then $f_\pm \in C_c(G)$, $f_\pm \geq 0$,

and $f = f_+ - f_-$, so that

$$\left|\int_G f\right| = \left|\int_G f_+ - \int_G f_-\right|$$

$$\leq \left|\int_G f_+\right| + \left|\int_G f_-\right|$$

$$= \int_G f_+ + \int_G f_- \qquad \text{[nonnegative]}$$

$$= \int_G |f| . \qquad \square \qquad [f_+ + f_- = |f|]$$

We now seek a special type of integral on G that plays well with translation:

Defn For $g \in G$, $f \in C_c(G)$, define

$$L_g f : G \longrightarrow \mathbb{C}$$
$$x \longmapsto f(g^{-1}x)$$

the left translation of f by g.

Note $L_g f \in C_c(G)$ as well, and

$$L_g(L_h f) = L_{gh} f, \quad L_e f = f$$

Q Why $f(g^{-1}x)$ and not $f(gx)$?

- Want $(L_g f)(g) = f(e)$
- $(L_a f)(x) = f(-a + x)$ for $G = (\mathbb{R}, +)$

so this is a left action of G on $C_c(G)$.

Defn An integral $I: C_c(G) \longrightarrow \mathbb{C}$ is (left) invariant (or Haar) when $I(L_g f) = I(f)$ $\forall f \in C_c(G), g \in G$.

(Equivalently, $\int_G f(gx)\,dx = \int_G f(x)\,dx$ $\forall g \in G, f \in C_c(G)$.)

Eg. The Riemann integral $\int_{-\infty}^{\infty} f(x)\,dx$ is left invariant for $(\mathbb{R}, +)$.

Exercise Show that $f \longmapsto \int_0^{\infty} \frac{f(x)}{x}\,dx$ is a Haar integral for $(\mathbb{R}_{>0}, \cdot)$.

Need: · linearity · nonnegativity · left-invariance

For left-invariance, $a>0$ $\quad L_a f \longmapsto \int_0^\infty \frac{f(a^{-1}x)}{x}\,dx = \int_0^\infty \frac{f(u)}{\cancel{a}u}\,\cancel{a}\,du$

$u = a^{-1}x$

$du = a^{-1}\,dx$

$$= \int_0^\infty \frac{f(u)}{u}\,du \quad \checkmark$$

Thm There exists a nontrivial Haar integral I for G.

If I' is a second Haar integral, then $\exists c \geq 0$ s.t. $I' = cI$.

Pf Appendix B of Deitmar. $\square$

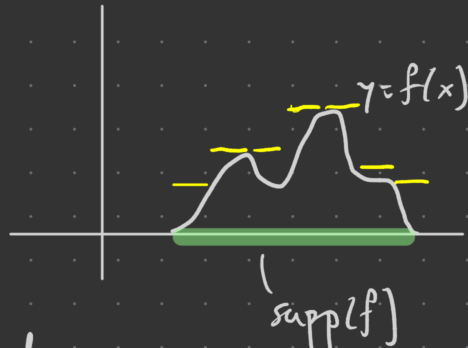
Construction

First recall the Riemann integral for $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ cts, cptly supp'd.

For $n \in \mathbb{Z}_{\geq 1}$, let $\chi_n := \chi_{\left[-\frac{1}{2n}, \frac{1}{2n}\right]}$. There exist $x_1, \ldots, x_m \in \mathbb{R}$

$c_1, \ldots, c_m > 0$ such that

$$f(x) \leq \sum_{j=1}^{m} c_j \chi_n(x - x_j)$$



Define $$(f : \chi_n) := \inf \left\{ \sum_{j=1}^{m} c_j \;\middle|\; \begin{array}{l} c_1, \ldots, c_m > 0 \text{ and} \\ \exists\, x_1, \ldots, x_m \in \mathbb{R} \text{ s.t. } f(x) \leq \sum_{j=1}^{m} c_j \chi_n(x - x_j) \end{array} \right\}$$

Then $$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{n \to \infty} \frac{1}{n} (f : \chi_n).$$

Note that if $f_0 = \chi_{[0,1]}$, then $(f_0 : x_n) = n$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n \to \infty} \frac{(f : x_n)}{(f_0 : x_n)}.$$

This generalizes: Fix $f_0 : G \to \mathbb{R}_{\geq 0}$ nonzero.

Set $$\int_G f(x)\,dx = \lim_{U \to \{e\}} \frac{(f : x_U)}{(f_0 : x_U)}$$ for $f : G \to \mathbb{R}_{\geq 0} \in C_c(G)$

where: U = open nbhd of $e \in G$ shrinking to $\{e\}$.

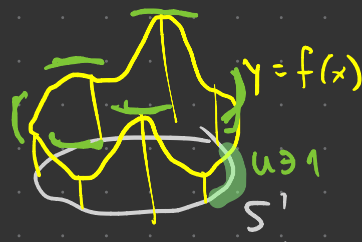
$$(f : \chi_U) := \inf \left\{ \sum_{j=1}^{m} c_j \;\middle|\; \begin{array}{l} c_1, \dots, c_m > 0 \text{ and } \exists g_1, \dots, g_m \in G \text{ s.t.} \\ f(x) \le \sum_{j=1}^{m} c_j L_{g_j} \chi_U(x) \end{array} \right\}$$

Fact This is a nontrivial left-inv't integral.

Fact $(C_c(G), \langle , \rangle)$ is an inner product space with

$$\langle f_0, f_1 \rangle = \int_G f_0 \cdot \overline{f_1}. \qquad \left(\text{for } \int_G \text{ Haar integration}\right)$$

Defn The Hilbert space completion of $C_c(G)$ is called $L^2(G)$.



$(f : X_u)$ optimizes trans'ns, heights to make this small

$$\lim_{u \to \{e\}} F(u) = L \quad \text{means}$$

$\forall \varepsilon > 0 \ \exists U$ nbhd of e s.t.

$$|F(U) - L| < \varepsilon .$$