Henceforth, A is a locally compact Hausdorff topological group; shorthand LCA group. (the H is silent)

Defn The Pontryagin dual of A is

$$\hat{A} := \operatorname{Hom}_{\text{cts}}(A, S^1)$$

$$\begin{array}{ccc} \mathbb{C}^\times & = & GL_1\mathbb{C} \\ \vee\!\!\mid & & \vee\!\!\mid \\ S^1 & = & U_1(\mathbb{C}) \end{array}$$

the group of unitary characters of A under pointwise multiplication.

We endow $\hat{A}$ with the compact open topology: the coarsest topology with opens $P(K, U) := \{\chi \in \hat{A} \mid \chi K \subseteq U\}$ for all $K \subseteq A$ compact, $U \subseteq S^1$ open.

Note Opens of $\hat{A}$ = unions of finite intersections of $P(K_i, U_i)$.

E.g. $\hat{S^1} \cong \mathbb{Z}$ w/ discrete topology

$\hat{\mathbb{R}} \cong \mathbb{R}$ w/ standard topology

$\hat{A} \cong A$ for A finite discrete.

Prop If A is an LCA group, then $\hat{A}$ is an LCA group.

Pf For local compactness Suffices to show $1 \in \hat{A}$ has a compact neighborhood.

Let K be a compact nbhd of e in A,

$$U := e^{2\pi i(-1/4, 1/4)}.$$

<u>Claim</u> $\overline{P(K,U)}$ is a compact nbhd of $1 \in \hat{A}$.

<u>Pf</u> HW.

Why is $\hat{A}$ Hausdorff?

Suffices to show $\{1\} \subseteq \hat{A}$ closed

$\iff \hat{A} \setminus 1$ open. But $\chi \neq 1 \Rightarrow \chi(a) \neq 1$ for some $a \in A \setminus e$.

Let $U = S^1 \setminus 1$. Then $\chi \in P(\{a\}, U)$ but $1 \notin P(\{a\}, U)$.

Thus $\hat{A} \setminus 1 = \bigcup_{a \neq e} P(\{a\}, U)$ is open, as desired.

Finally, we need to show multiplication, inversion are continuous.

We can do this all together by proving $f : \hat{A} \times \hat{A} \longrightarrow \hat{A}$
$$(\chi, \eta) \longmapsto \chi\eta^{-1}$$
is cts. (Why?) Moral exc: can be checked by showing $\chi_i, \eta_i \xrightarrow[\text{loc unif}]{} \chi, \eta \Rightarrow \chi_i \eta_i^{-1} \xrightarrow[\text{loc unif}]{\square} \chi\eta$

<u>Thm</u> For A an LCA group, A discrete $\Rightarrow \hat{A}$ compact
A compact $\Rightarrow \hat{A}$ discrete.

every subset is open $\Leftrightarrow$ all singletons open

<u>Pf</u> First suppose A compact. Then $\forall U \subseteq S^1$ open, $P(A, U) \subseteq \hat{A}$ is open.

For $U = e^{2\pi i(-1/2, 1/2)}$, $P(A, U) = \{1\}$, so $\{1\}$ is open $\Rightarrow \{\chi\}$ open $\forall \chi \in \hat{A}$. Thus $\hat{A}$ is discrete

$\chi(a) \neq 1 \Rightarrow \chi(a)^N \notin U$

$\Rightarrow \chi(aN) \notin U$

χO

Now suppose A is discrete. Idea: Use a compact exhaustion of A to show A is countable then prove $\hat{A}$ sequentially compact. $\square$

Thm (Pontryagin duality) For A an LCA group,

$$\mathrm{eval}: A \longrightarrow \hat{\hat{A}}$$

is an isomorphism of LCA groups.

Functoriality

Write LCA for the category of LCA gps + cts gp homomorphisms.

- composition = composition of functions is associative
- identity ✓

$\hat{(\ )} = \mathrm{LCA}(-, S^1) : \mathrm{LCA} \longrightarrow \mathrm{LCA}^{op}$

$$\begin{tikzcd} A \arrow[d, "f"'] & & \hat{A} & x \circ f \\ B & \arrow[r, mapsto] & \hat{B} \arrow[u, "f^*"] & x \arrow[u, mapsto] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f^* x"'] & B \arrow[d, "x"] \\ & S^1 \end{tikzcd}$$

satisfies $id_A^* = id_{\hat{A}}$ and $(g \circ f)^* = f^* \circ g^*$:

$$(g \circ f)^*(x) = x \circ (g \circ f) = (x \circ g) \circ f = f^*(g^* x)$$

Claim $\hat{(\ )}$ is an anti-equivalence of categories.

$F : C \longrightarrow D$ is an equiv of cats when $\exists\, G : D \longrightarrow C$ s.t. $\alpha : G \circ F \overset{\cong}{\Longrightarrow} \mathrm{id}_C$ and $\beta : F \circ G \overset{\cong}{\Longrightarrow} \mathrm{id}_D$.

$$\begin{tikzcd} c \arrow[d, "f"'] \\ c' \end{tikzcd} \qquad \begin{tikzcd} GFc \arrow[r, "\alpha_c", "\cong"'] \arrow[d, "GFf"'] \arrow[dr, phantom, "\circlearrowright"] & c \arrow[d, "f"] \\ GFc' \arrow[r, "\cong", "\alpha_{c'}"'] & c' \end{tikzcd}$$

For $\hat{(\,)} : LCA \xrightarrow{\simeq} LCA^{op}$ need $\hat{\hat{(\,)}} \cong \mathrm{id}_{LCA}$

$$\begin{tikzcd} A \arrow[r, "\mathrm{eval}"] \arrow[d, "f"'] & \hat{\hat{A}} \arrow[d, "f^{**}"] \\ B \arrow[r, "\mathrm{eval}"'] & \hat{\hat{B}} \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{eval}_a \arrow[d, mapsto] \\ f(a) \arrow[r, mapsto] & \mathrm{eval}_{f(a)} \end{tikzcd}$$

$$\begin{tikzcd} \hat{B} \arrow[r, "f^*"] \arrow[dr] & \hat{A} \arrow[d, "x"] \\ & S^1 \end{tikzcd}$$

$$f^{**}\left(\begin{array}{c}\hat{A}\\ \downarrow x\\ S^1\end{array}\right) = x \circ f^*$$

$= \mathrm{eval}_a$

$\eta \longmapsto \eta \circ f$

$\downarrow$

$x(\eta \circ f)$

$= (\eta \circ f)(a)$

$= \eta(f(a))$

$= \mathrm{eval}_{f(a)}(\eta)$

E.g. Consider $A = \mu_{p^\infty} := \{ z \in S^1 \mid z^{p^n} = 1 \text{ for some } n \in \mathbb{N} \}$, p prime.

This is the Prüfer p-group. What topology shall we give it?

– w/ discrete top!

$\star \quad \mu_p \hookrightarrow \mu_{p^2} \hookrightarrow \mu_{p^3} \hookrightarrow \cdots$

with union = colimit μ_{p^∞} :

$$\mu_{p^\infty} = \operatorname*{colim}_n \mu_{p^n} .$$

Hit $\star$ with $\widehat{(\ )}$:

$$\begin{array}{ccccccc} \widehat{\mu}_p & \longleftarrow & \widehat{\mu}_{p^2} & \longleftarrow & \widehat{\mu}_{p^3} & \longleftarrow & \cdots \\ \wr\| & & \wr\| & & \wr\| & & \\ \mathbb{Z}/p\mathbb{Z} & \longleftarrow & \mathbb{Z}/p^2\mathbb{Z} & \longleftarrow & \mathbb{Z}/p^3\mathbb{Z} & \longleftarrow & \cdots \end{array}$$

Fact Representable functors take colimits to limits so

$$\widehat{\mu_{p^\infty}} = \widehat{\operatorname{colim}_n \mu_{p^n}} = \lim_n \widehat{\mu_{p^n}} = \lim_n \mathbb{Z}/p^n\mathbb{Z} =: \mathbb{Z}_p$$

p-adic integers.

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Category Theory in Context