$f : A \to \mathbb{C}$ s.t. $\sum_{a \in A} |f(a)|^2 < \infty$ automatic since A finite

$\ell^2(A) = \mathbb{C}^A$ contains all characters — inner prod space via

$\{f \mid f : A \to \mathbb{C}\}$

$$\langle f, g \rangle := \sum_{a \in A} f(a)\overline{g(a)}$$

Lemma For $\chi, \eta \in \hat{A}$, $\langle \chi, \eta \rangle = \begin{cases} |A| & \text{if } \chi = \eta \\ 0 & \text{o/w} \end{cases}$.

slogan: characters are orthogonal

Pf If $\chi = \eta$, then $\langle \chi, \eta \rangle = \sum_{a \in A} \chi(a)\overline{\eta(a)} = \sum_{a \in A} \chi(a)\overline{\chi(a)} = \sum_{a \in A} 1 = |A|$

For $\chi \neq \eta$, let $\alpha = \chi\eta^{-1} \neq 1 \in \hat{A}$. Then

$$\langle \chi, \eta \rangle = \sum_{a \in A} \chi(a)\eta(a)^{-1} = \sum_{a \in A} \alpha(a).$$

Take $b \in A$ with $\alpha(b) \neq 1$. Then

$$\langle \chi, \eta \rangle \alpha(b) = \sum_{a \in A} \alpha(a)\alpha(b) = \sum_{a \in A} \alpha(ab) = \langle \chi, \eta \rangle$$

$\{ab \mid a \in A\} = A$ (exc)

so $\langle \chi, \eta \rangle = 0$. $\square$

Defn For $f \in \ell^2(A)$, define its Fourier transform $\hat{f} : \hat{A} \to \mathbb{C}$ by

$$\hat{f}(\chi) = \frac{1}{\sqrt{|A|}} \langle f, \chi \rangle = \frac{1}{\sqrt{|A|}} \sum_{a \in A} f(a) \overline{\chi(a)} .$$

needed b/c $\hat{A}$ is only orthogonal, not orthonormal.

Thm $\ell^2(A) \xrightarrow{\cong} \ell^2(\hat{A})$ and $\hat{\hat{f}}(\mathrm{eval}_a) = f(a^{-1})$.

$f \longmapsto \hat{f}$

so similar to $\hat{\hat{f}}(x) = f(-x)$ for $f \in L^1_{loc}(\mathbb{R})$.