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Fourier analysis on finite Abelian groups

Let A be a finite Abelian group.

<u>E.g.</u> $A = S^1, \mathbb{R}$ but <u>not</u> finite

- $\mathbb{Z}/5\mathbb{Z}$, other cyclic groups $C_n = \langle x \mid x^n \rangle \cong \{ e^{2\pi i k/n} \mid k = 0, 1, \ldots, n-1 \}$
- $(\mathbb{Z} \setminus \{0\}, \cdot)$ is a commutative monoid but no inverses & not finite
  $(\{\pm 1\}, \cdot)$ is a finite group $\cong C_2$
- $C_2 \times C_2 = K_4$ Klein 4-group

<u>Thm</u> Every finite Abelian group is isomorphic to a product of cyclic groups.

Defn A character of A is a homomorphism $\chi : A \to \mathbb{C}^\times$.

$$\mathbb{C}^\times = (\mathbb{C}\setminus\{0\}, \cdot)$$

Prop $\operatorname{im}\chi \le S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = U(1)$

Pf For each $a \in A$, $\exists n \in \mathbb{Z}_{>0}$ s.t. $a^n = 1$ (id of A). (In fact, $n \mid |A|$.)

Thus $\underset{\mathbb{C}}{1} = \chi(1) = \chi(a^n) = \chi(a)^n \implies \underset{\in S^1,}{\chi(a) = e^{2\pi i k/n}}$ for some $k \in \mathbb{Z}$

Rmk In fact $\operatorname{im}(\chi) \le \mu_\infty = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}\} \cong \mathbb{Q}/\mathbb{Z}$ $\square$

Write $\hat{A} := \{\text{characters of } A\}$. Equipped with pointwise product

$$\chi\eta(a) = \chi(a)\eta(a)$$

this is the Pontryagin dual of A.

- id of $\hat{A}$ is $1 : A \to \mathbb{C}^\times$, $a \mapsto 1$
- $\chi^{-1} = \bar{\chi}$: $\bar{\chi}(ab) = \overline{\chi(ab)} = \overline{\chi(a)\chi(b)} = \bar{\chi}(a)\bar{\chi}(b)$ ✓

Note $\hat{A}$ is Abelian.

Q What are the characters of C_n?
$\quad\quad\quad\quad \langle x \mid x^n \rangle$

A $\chi : C_n \longrightarrow \mathbb{C}^\times$ specified by $\chi(x)$ + $\chi(x^k) = \chi(x)^k$

but – as before – $\chi(x) = e^{2\pi i l/n}$ for some $l \in \mathbb{Z}$ or $l = 0, 1, \ldots, n-1$

Define $\chi_l \in \widehat{C_n}$ to be the unique such character.

Then $\chi_l \chi_m (x) = \chi_l(x)\chi_m(x) = e^{2\pi i l/n} e^{2\pi i m/n}$

$$= \chi_{\underbrace{l+m}_{\text{in } \mathbb{Z}/n\mathbb{Z}}}(x)$$

$z^n = 1$

$z^k \neq 1,\ 1 \leq k \leq n-1$

Hence $\widehat{C_n} \cong C_n$.

$\chi_1 \mapsfrom x$

or χ_l for $\gcd(l, n) = 1$

Thm There is a canonical isomorphism

$$\begin{array}{ccccc} A & \xrightarrow{\cong} & \hat{\hat{A}} & & \\ a & \longmapsto & \hat{A} & & \chi \\ & & \downarrow \mathrm{eval}_a & & \downarrow \\ & & \mathbb{C}^\times & & \chi(a) \end{array}$$

.

Pf $\mathrm{eval}_{ab}(\chi) = \chi(ab) = \chi(a)\chi(b) = \mathrm{eval}_a(\chi)\,\mathrm{eval}_b(\chi)$ so eval is a homomorphism.

Now proceed by induction:

(a) true for cyclic groups

(b) true for $A, B \Rightarrow$ true for $A \times B$.

(a) Suppose $A = C_n$. Since $\hat{C}_n \cong C_n$, also $\hat{\hat{C}}_n \cong C_n$.
In particular, $|\hat{\hat{A}}| = |A|$. Thus it suffices to show

$\mathrm{eval}: A \longrightarrow \hat{\hat{A}}$ is injective $\iff$ $\ker(\mathrm{eval}) = 1$.

Suppose $\mathrm{eval}_a = 1 : \hat{A} \longrightarrow \mathbb{C}^\times$ So $a \in \ker(\mathrm{eval})$

$x \longmapsto 1 = x(a)$

iff $x(a) = 1 \;\; \forall x \in \hat{A}$. Since $A = C_n$, each $x \in \hat{A}$

is of the form $x_l : x \longmapsto e^{2\pi i l/n}$. Write $a = x^k \in C_n$.

Then $1 = x_l(a) = e^{2\pi i k l/n} \Rightarrow n \mid k \Rightarrow a = 1$.

Hence eval is injective $\Rightarrow$ isomorphism.

(b) Follows from $\widehat{A \times B} \cong \hat{A} \times \hat{B}$ — exc to finish. $\square$

$$\begin{array}{ccc} A\times B & & A \qquad\quad B \\ x \downarrow & \longmapsto & x(-,1)\downarrow \;,\; \downarrow x(1,-) \\ \mathbb{C}^\times & & \mathbb{C}^\times \qquad \mathbb{C}^\times \end{array}$$