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Planchrel's Theorem

Let $L^2_{bc}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ bdd cts with } \underbrace{\int_{\mathbb{R}} |f|^2}_{=: \|f\|_2^2} < \infty \}$.

Lemma $L^2_{bc}(\mathbb{R})$ is an inner product space with

$$\langle f, g \rangle = \int_{\mathbb{R}} f \cdot \bar{g} \,,$$

and $L^1_{bc}(\mathbb{R}) \leq L^2_{bc}(\mathbb{R})$. $\square$

Thm [Planchrel] For every $f \in L^1_{bc}(\mathbb{R})$, we have $\hat{f} \in L^2_{bc}(\mathbb{R})$ and

$$\|f\|_2 = \|\hat{f}\|_2 \,.$$

Pf Let $\overleftarrow{f}(x) = f(-x)$ and set $g := \overleftarrow{f} * f$. Then

$$g(x) = \int_{\mathbb{R}} f(t-x) f(t)\, dt$$

and $$g(0) = \|f\|_2^2 .$$

Now $\hat{g}(t) = \hat{\overleftarrow{f}}(t)\hat{f}(t) = \overline{\hat{f}(t)}\, \hat{f}(t) = |\hat{f}(t)|^2$. Thus

$$\|f\|_2^2 = g(0) = \lim_{\lambda \to 0} g * h_\lambda(0)$$

$$= \lim_{\lambda \to 0} \int_{\mathbb{R}} e^{-\lambda|t|} \hat{g}(t)\, dt$$

$$= \lim_{\lambda \to 0} \int_{\mathbb{R}} e^{-\lambda|t|} |\hat{f}(t)|^2\, dt$$

$$= \|\hat{f}\|_2^2 \qquad \text{[monotone convergence]} \qquad \square$$

Upshot $\hat{()} : L^1_{bc}(\mathbb{R}) \longrightarrow L^2_{bc}(\mathbb{R})$ is an isometric embedding of $(L^1_{bc}(\mathbb{R}), \|\cdot\|_2)$ into $L^2_{bc}(\mathbb{R})$.

i.e. $\hat{()}$ is unitary: $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$.

Poisson Summation

Thought experiment: suppose $f : \mathbb{R} \to \mathbb{C}$ cts and $\forall x \in \mathbb{R}$

$$g(x) := \sum_{m \in \mathbb{Z}} f(x+m) \qquad\qquad g(x+1) = \underbrace{\sum_{m \in \mathbb{Z}} f(x+1+m)}_{} = g(x)$$

converges absolutely. Then $g : \mathbb{R} \to \mathbb{C}$ is 1-periodic.

Assume the Fourier series of g converges pointwise to g so that

$$g(x) = \sum_{n \in \mathbb{Z}} \hat{g}(n) e^{2\pi i n x}.$$

Then for $x = 0$,

$$\sum_{m \in \mathbb{Z}} f(m) = g(0)$$

$$= \sum_{n \in \mathbb{Z}} \hat{g}(n)$$

$$= \sum_{n \in \mathbb{Z}} \int_0^1 \Big(\underbrace{\sum_{m \in \mathbb{Z}} f(y+m)}_{g(y)} e^{-2\pi i n y} \Big)\, dy$$

Suppose $\int \longleftrightarrow \sum$ swap legally. Then

$$(\star) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(y) e^{-2\pi i n y} \, dy$$

$$= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} f(y) e^{-2\pi i n y} \, dy$$

$$= \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

$\hat{g}(n)$ = n-th Fourier coeff of g

$\hat{f}(n)$ = Fourier transform of f evaluated at n.

So guess: $\sum_{k \in \mathbb{Z}} f(k) = \sum_{k \in \mathbb{Z}} \hat{f}(k)$.

Poisson summation

Thm Let $f \in L^1_{loc}(\mathbb{R})$ be piecewise ctsly differentiable with finitely many exceptions. Let $\varphi(x) = \begin{cases} f'(x) & \text{if it exists} \\ 0 & \text{o/w} \end{cases}$.

Suppose $x^2 f(x)$, $x^2 \varphi(x)$ are bounded. Then

$$\sum_{k \in \mathbb{Z}} f(k) = \sum_{k \in \mathbb{Z}} \hat{f}(k).$$

E.g. $f \in \mathcal{S}$

Pf Let $g(x) = \sum_{k \in \mathbb{Z}} f(x+k)$. Since $x^2 f(x) \leq C$ for some constant C, we know $|f(x+k)| \leq \frac{C}{|x+k|^2}$ so g converges uniformly and absolutely. The same is true of $\tilde{g}(x) = \sum_{k \in \mathbb{Z}} \varphi(x+k)$.

We aim to show g is piecewise ctsly diff'l $\Rightarrow$ pointwise convergence of Fourier series.

We have
$$\int_0^x \tilde{g}(t)\,dt = \int_0^x \sum_{k\in\mathbb{Z}} \varphi(t+k)\,dt$$
$$= \sum_{k\in\mathbb{Z}} \int_0^x \varphi(t+k)\,dt \qquad \text{[sum converges unif]}$$
$$= \sum_{k\in\mathbb{Z}} \int_k^{k+x} \varphi(t)\,dt$$
$$= \sum_{k\in\mathbb{Z}} f(k+x) - f(k) \qquad \text{[FTC2]}$$
$$= g(x) - g(0).$$

I.e. $g(x) = g(0) + \int_0^x \tilde{g}(t)\,dt$ is piecewise ctsly diff'l, as desired.

Justify $\dagger$ by unif conv of sum on $[0,1]$. $\square$

Theta Series

For $t > 0$, let $\Theta(t) := \sum_{k \in \mathbb{Z}} e^{-t\pi k^2}$.

Thm For all $t > 0$, $\Theta(t) = t^{-1/2}\,\Theta(1/t)$.

Pf Set $f_t(x) := e^{-t\pi x^2}$. We have shown $\hat{f}_1 = f_1$, and since $f_t(x) = f_1(\sqrt{t}\,x)$, we get

$$\hat{f}_t(x) = t^{-1/2}\hat{f}_1\left(\frac{x}{\sqrt{t}}\right) = t^{-1/2} f_1\left(x/\sqrt{t}\right)$$

$$= t^{-1/2} f_{\frac{1}{t}}(x).$$

Since $f_t \in \mathcal{S}$, Poisson summation applies:

$$\Theta(t) = \sum_{k\in\mathbb{Z}} f_t(k) = \sum_{k\in\mathbb{Z}} \hat{f}_t(k).$$

$$= t^{-1/2} \sum_{k\in\mathbb{Z}} f_{\frac{1}{t}}(k)$$

$$= t^{-1/2}\,\Theta\left(\tfrac{1}{t}\right). \quad \square$$

Why care? Can "extend" Θ to a function on $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\}$.

Write Θ as a series in $q = e^{\pi i z}$:

$$\Theta(z) = \sum a_k q^k \quad \text{s.t.} \quad a_k = \begin{cases} 2 & k \text{ is square } \neq 0 \\ 1 & k = 0 \\ 0 & \text{o/w}. \end{cases}$$

"q series"

Use the trans'n property $\Theta(t) = t^{-1/2}\,\Theta\left(\frac{1}{t}\right)$ to prove Θ^4 is a modular form — has weight 2 and can write as a linear combo of Eisenstein forms

$\rightsquigarrow$ explicit fmla for q-series of Θ^4.

$$\Theta^4 = \left(\sum a_k q^k\right)^4$$

$$= \sum b_k q^k \quad \text{with } b_k \geq 1 \text{ iff } k = n_1^2 + n_2^2 + n_3^2 + n_4^2.$$

Explicit fmla: ① $b_k \geq 1 \ \forall k$ — Lagrange's 4 squares theorem

② explicit values of $b_k = \cdots$.

Mumford, Tata Lectures on Theta Series | Diamond–Shurman