2025.III.3

Inversion Formula

Defn For $\lambda > 0$, the Gauss kernel is

$$h_\lambda : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i t x} \, dt.$$

Lemma 1 In fact, $h_\lambda(x) = \dfrac{2\lambda}{4\pi^2 x^2 + \lambda^2}$ and $\int_{\mathbb{R}} h_\lambda(x)\, dx = 1$.

It follows that for $\lambda > 0$, $h_\lambda(x) = \frac{1}{\lambda} h_1\left(\frac{x}{\lambda}\right)$.

Pf We compute

$$h_\lambda(x) = \int_0^\infty e^{2\pi i t x - \lambda t}\,dt + \int_{-\infty}^0 e^{2\pi i t x + \lambda t}\,dt$$

$$= \left.\frac{e^{2\pi i t x - \lambda t}}{2\pi i x - \lambda}\right|_0^\infty + \left.\frac{e^{2\pi i t x + \lambda t}}{2\pi i x + \lambda}\right|_{-\infty}^0$$

$$= \frac{1}{\lambda - 2\pi i x} + \frac{1}{\lambda + 2\pi i x}$$

$$= \frac{2\lambda}{\lambda^2 + 4\pi^2 x^2}.$$

Further,

$$\int_{\mathbb{R}} h_\lambda(x)\,dx = \frac{2}{\lambda}\int_{\mathbb{R}} \frac{1}{1+\left(\frac{2\pi x}{\lambda}\right)^2}\,dx \qquad u = \frac{2\pi x}{\lambda} \qquad du = \frac{2\pi}{\lambda}\,dx$$

$$= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{1+u^2}\,du$$

$$= \frac{1}{\pi}\arctan(u)\Big|_{-\infty}^{\infty}$$

$$= 1. \qquad \square$$

Lemma 2 If $f \in L^1_{bc}(\mathbb{R})$, then for all $\lambda > 0$,

$$f * h_\lambda(x) = \int_{\mathbb{R}} e^{-\lambda|t|} \hat{f}(t) e^{2\pi i x t} \, dt.$$

Pf Let's compute:

$$f * h_\lambda(x) = \int_{\mathbb{R}} f(x-y) h_\lambda(y) \, dy$$

$$= \int_{\mathbb{R}} f(x-y) \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i t y} \, dt \, dy$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} \int_{\mathbb{R}} f(x-y) e^{2\pi i t y} \, dy \, dt \qquad y \leftarrow x-y$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} \int_{\mathbb{R}} f(y) e^{2\pi i t(x-y)} \, dy \, dt$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i x t} \int_{\mathbb{R}} f(y) e^{-2\pi i t y} \, dy \, dt$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i x t} \hat{f}(t) \, dt. \qquad \square$$

Lemma 3 For all $f \in L^1_{loc}(\mathbb{R})$, $x \in \mathbb{R}$,

$$\lim_{\lambda \to 0^+} f * h_\lambda(x) = f(x).$$

Pf Again, calculate :

$$f * h_\lambda(x) - f(x) = \int_{\mathbb{R}} f(x-y)\, h_\lambda(y)\, dy - f(x)$$

so $f(x) = \int_{\mathbb{R}} f(x) h_\lambda(y)\, dy$

$$= \int_{\mathbb{R}} (f(x-y) - f(x))\, h_\lambda(y)\, dy \qquad \left[\int_{\mathbb{R}} h_\lambda = 1\right]$$

$$= \int_{\mathbb{R}} (f(x-y) - f(x))\, \tfrac{1}{\lambda} h_1(y/\lambda)\, dy$$

$u = \frac{y}{\lambda}$, $du = \frac{1}{\lambda}\, dy$

$$= \int_{\mathbb{R}} (f(x-\lambda u) - f(x))\, h_1(u)\, du .$$

Since $f \in L^1(\mathbb{R})$, $\exists\, C>0$ s.t. $|f(x)| \le C$ $\forall x \in \mathbb{R}$.

Thus the integrand is dominated by $2C h_1(u)$.

As $\lambda \to 0^+$, $f(x-\lambda u) \longrightarrow f(x)$ locally uniformly in u.

So by dominated convergence,

$$f * h_\lambda(x) - f(x) \xrightarrow[\lambda \to 0^+]{} 0. \qquad \square$$

Thm [inversion formula] Let $f \in L^1_{bc}(\mathbb{R})$ and assume $\hat{f} \in L^1_{bc}(\mathbb{R})$ as well. Then $\forall x \in \mathbb{R}$,

$$\boxed{\hat{\hat{f}}(x) = f(-x)}.$$

Cor $\hat{\hat{\hat{\hat{f}}}} = f$

Cor

=

Pf The claim is equivalent to

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i x y} \, dy.$$

For $\lambda > 0$,

$$f * h_\lambda (x) = \int_{\mathbb{R}} e^{-\lambda|t|} \hat{f}(t) e^{2\pi i x t} \, dt$$

by Lemma 2. As $\lambda \to 0^+$, the LHS $\longrightarrow f(x)$ by Lemma 3.

The RHS integrand is dominated by $|\hat{f}(t)|$, so by dominated convergence,

$$f(x) = \lim_{\lambda \to 0^+} \int_{\mathbb{R}} e^{-\lambda|t|} \hat{f}(t) e^{2\pi i x t} \, dt = \int_{\mathbb{R}} \hat{f}(t) e^{2\pi i x t} \, dt. \quad \square$$

Cor $\hat{()} : \mathcal{S} \xrightarrow{\cong} \mathcal{S}$ with characteristic polynomial $X^4 - 1$.

$\Longrightarrow$ the only eigenvalues of $\hat{()}$ are $\pm 1, \pm i$.

Prop Let $f(x) = e^{-\pi x^2}$. Then $f \in \mathcal{S}$ and $\hat{f} = f$.

Pf Observe $f'(x) = -2\pi x e^{-\pi x^2} = -2\pi x f(x)$.

In fact (exc), f is the unique sol'n to this diff'l eqn up to scalar multiplication. We have $f \in \mathcal{S}$ (why?) so $\hat{f} \in \mathcal{S}$ and we can compute

$$(\hat{f})'(y) = \int_{\mathbb{R}} (-2\pi i x) e^{-\pi x^2} e^{-2\pi i x y} \, dx$$

$$= i \int_{\mathbb{R}} (e^{-\pi x^2})' e^{-2\pi i x y} \, dx$$

$u = e^{-2\pi i x y}$ $\quad dv = (e^{-\pi x^2})' \, dx$

$du = -2\pi i y e^{-2\pi i x y} \, dx$ $\quad v = e^{-\pi x^2}$

$$= \cancelto{0}{i \, uv \Big|_{-\infty}^{\infty}} - 2\pi y \int_{\mathbb{R}} f(x) e^{-2\pi i x y} \, dx$$

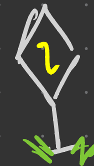
$$= -2\pi y \, \hat{f}(y).$$

By the diff'l eqn, $\hat{f}(y) = c e^{-\pi^2 y}$ for some constant c.

Since $\hat{\hat{f}}(x) = f(-x)$, know $c^2 = 1 \Rightarrow c = \pm 1$. Since

$\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2} \, dx > 0$, we must have $c = 1$. $\square$

Cor $\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$.



No elementary expression for anti-derivative of e^{-x^2}

Pf Let $f(x) = e^{-\pi x^2}$. By the proposition,

$$1 = f(0) = \hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \cdot 0}\, dx$$

$$= \int_{\mathbb{R}} e^{-\pi x^2}\, dx.$$

Now $\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi} \int_{\mathbb{R}} e^{-\pi u^2}\, du = \sqrt{\pi}$. $\square$

$u = \frac{x}{\sqrt{\pi}}$

$du = \frac{1}{\sqrt{\pi}}\, dx$