2025. II. 25

Defn For $f \in L^1_{bc}(\mathbb{R})$, the Fourier transform of f is

$$\begin{aligned} \hat{f} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto \hat{f}(\gamma) := \int_{\mathbb{R}} f(x) e^{-2\pi i \gamma x} \, dx . \end{aligned}$$

Thm Let $f \in L^1_{bc}(\mathbb{R})$.

(a) If $g(x) = f(x) e^{2\pi i a x}$, then $\hat{g}(\gamma) = \hat{f}(\gamma - a)$ ✓

(b) If $g(x) = f(x-a)$, then $\hat{g}(\gamma) = \hat{f}(\gamma) e^{-2\pi i a \gamma}$ ✓

(c) If $g \in L^1_{bc}(\mathbb{R})$ and $h = f * g$, then $\hat{h}(\gamma) = \hat{f}(\gamma) \hat{g}(\gamma)$

(d) If $g(x) = f\left(\frac{x}{\lambda}\right)$ for $\lambda > 0$, then $\hat{g}(\gamma) = \lambda \hat{f}(\lambda \gamma)$ ✓

(e) If $g(x) = -2\pi i x f(x)$, $g \in L^1_{bc}(\mathbb{R})$, then $\hat{f} \in C^1(\mathbb{R})$ with

$$\hat{f}'(\gamma) = \hat{g}(\gamma).$$

(f) Suppose $f \in C^1(\mathbb{R})$ and $f, f' \in L^1_{bc}(\mathbb{R})$. Then $\widehat{f'}(\gamma) = 2\pi i \gamma \hat{f}(\gamma)$; in particular, $\gamma \mapsto \gamma \hat{f}(\gamma)$ is bounded.

$|\gamma \hat{f}(\gamma)| \leq C$ for some $C > 0$

$\Rightarrow |\hat{f}(\gamma)| \leq \frac{C}{|\gamma|} \to 0$ $\gamma \to \pm\infty$.

(g) Suppose $f \in C^2(\mathbb{R})$ and $f, f', f'' \in L^1_{bc}(\mathbb{R})$. Then $\hat{f} \in L^1_{bc}(\mathbb{R})$.

Pf (c) We compute

$$\hat{h}(\gamma) = \int_{\mathbb{R}} h(x) e^{-2\pi i \gamma x} \, dx \qquad \text{defn}$$

$$= \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} f(x-t) g(t) \, dt}_{f * g\,(x)} \, e^{-2\pi i \gamma x} \, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) e^{-2\pi i \gamma x} \, dx \; g(t) \, dt$$

$u = x - t \Rightarrow x = u + t, du = dx$

$e^{-2\pi i \gamma x} = e^{-2\pi i \gamma u} e^{-2\pi i \gamma t}$

$$= \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} f(u) e^{-2\pi i \gamma u} \, du}_{\hat{f}(\gamma)} \; g(t) e^{-2\pi i \gamma t} \, dt$$

$$= \hat{f}(\gamma) \hat{g}(\gamma). \quad \square$$

Pf (c) Observe that $\dfrac{\hat{f}(\gamma) - \hat{f}(t)}{\gamma - t} = \displaystyle\int_{\mathbb{R}} f(x) e^{-2\pi i t x} \, \frac{e^{2\pi i (t-\gamma) x} - 1}{\gamma - t} \, dx.$

$e^{-2\pi i \gamma x}$

$u = \gamma - t \longrightarrow 0 \iff t \to \gamma$

Let $\varphi(x,u) = \dfrac{e^{-2\pi i u x} - 1}{u}$. Then $|\varphi(x,u)| \leq 2\pi |x|$ for $u \neq 0$ (exc)

and $\varphi(x,u) \longrightarrow -2\pi i x$ as $u \to 0$. The convergence is locally

$\text{"}\frac{\partial}{\partial u} \varphi(x,u)\Big|_{u=0}\text{"}$

uniform in x. The claim now follows from dominated convergence.

$\square$

Pf (f) Since $f \in L^1(\mathbb{R})$, $\exists\ (s_n), (t_n) \longrightarrow \infty$ with $f(-s_n), f(t_n) \to 0$.

Then $\widehat{f'}(\gamma) = \lim_{n \to \infty} \int_{-s_n}^{t_n} f'(x) e^{-2\pi i \gamma x}\, dx$

$u = e^{-2\pi i \gamma x}$ $dv = f'(x)\,dx$

$du = -2\pi i \gamma e^{-2\pi i \gamma x}$ $v = f(x)$

$$= \lim_{n \to \infty} \left(f(x) e^{-2\pi i \gamma x} \Big|_{-s_n}^{t_n} + 2\pi i \gamma \int_{-s_n}^{t_n} f(x) e^{-2\pi i \gamma x}\, dx \right)$$

$$= 2\pi i \gamma \hat{f}(\gamma). \quad \square$$

Pf (g) Applying (f) twice shows $\gamma^2 \hat{f}(\gamma)$ is bounded, so

$|\hat{f}(\gamma)| \leq \frac{C}{|\gamma|^2}$ for some $C \geq 0$. Since $\hat{f}$ is cts its integrable

so $\int_{\mathbb{R}} |\hat{f}| \leq \int_{\mathbb{R}} \frac{C}{|\gamma|^2} \, d\gamma < \infty$ by comparison test. $\square$

Schwartz space

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) \mid \sigma_{m,n}(f) < \infty \ \forall m,n \in \mathbb{N} \}$ where

$$\sigma_{m,n}(f) := \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty .$$

Q What do Schwartz functions "look like"?

A f (and all $f^{(n)}$) go to 0 at $\pm\infty$ faster than

$\frac{1}{\text{polynomial}}$. E.g. $f(x) = e^{-x^2}$.

Prop We have $\mathcal{S} \subseteq L^1_{bc}(\mathbb{R})$, and for $f \in \mathcal{S}$, $\hat{f} \in \mathcal{S}$ as well.

Pf Let $f \in \mathcal{S}$, so f is bdd cts and $\left|(1+x^2)f(x)\right| \leq C$ for some $C > 0$, so $\int_{\mathbb{R}} |f| \leq C \int_{\mathbb{R}} \frac{1}{1+x^2}\,dx = C\pi < \infty$.

$\left(\int \frac{1}{1+x^2}\,dx = \arctan(x) + C' \right)$

Thus $f \in L^1_{bc}(\mathbb{R})$.

By (e), $\hat{f} \in C^\infty(\mathbb{R})$ with $\hat{f}^{(n)}(x) = \widehat{(-2\pi i x)^n f(x)}$

By (f), $\widehat{f^{(n)}}(\gamma) = (2\pi i \gamma)^n \hat{f}(\gamma)$.

Thus each function $\gamma^m \hat{f}^{(n)}(\gamma)$ is a Fourier transform of some function in $\mathcal{S}$ and hence is bounded. $\square$

<u>Up next</u> inversion theorem for Fourier transforms:

$$\hat{\hat{f}}(x) = f(-x) \quad \text{for } f \in \mathcal{S}$$

Proof will use Gauss kernels:

for $\lambda > 0$, $x \in \mathbb{R}$

$$h_\lambda(x) := \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i t x} \, dt.$$