2025. II. 26

Fourier Transform

Idea For a non-periodic function $f : \mathbb{R} \to \mathbb{C}$, consider

$$\left.\begin{aligned} \hat{f} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} \, dx . \end{aligned}\right\} \text{Fourier transform of } f$$

To what extent can we recover f from $\hat{f}$?

Discussion To what extent is the Fourier transform similar to / different from Fourier series?

- $n \in \mathbb{N} \longleftrightarrow \gamma \in \mathbb{R}$
- $(\hat{f}(n))_{n \in \mathbb{Z}} \longleftrightarrow \hat{f} : \mathbb{R} \to \mathbb{C}$
- Still integrating f against e_γ

  but e_γ no longer integrable

  — not an inner product

Convergence Theorems

- Need to determine a good domain for $\hat{(\;)}$

Dominated convergence $(f_n : \mathbb{R} \to \mathbb{C})_n$ a sequence of functions converging locally uniformly to $f : \mathbb{R} \to \mathbb{C}$. Suppose $\exists g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\int_{\mathbb{R}} g < \infty$ and $|f_n(x)| \leq g(x)$ $\forall x \in \mathbb{R}$, $n \in \mathbb{N}$.

Then $\int_{\mathbb{R}} f_n$, $\int_{\mathbb{R}} f$ exist and $\lim_{n \to \infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$.

Pf [Deitmar, 3.1.1]. $\square$

"locally uniform" $\leftarrow$ $\forall x \in \mathbb{R}$ $\exists$ open nbhd $U \ni x$ on which convergence is uniform

Monotone Convergence $(f_n : \mathbb{R} \to \mathbb{R}_{\geq 0})_n$ a sequence of cts fns, $f_{n+1}(x) \geq f_n(x)$ $\forall n \in \mathbb{N}, x \in \mathbb{R}$, f a continuous fn s.t. $f_n \to f$ locally uniformly. Then $\lim_{n \to \infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$.

Pf [Deitmar 3.1.2]. $\square$

Convolution

Defn $L^1_{bc}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ bounded cts, } \int_{\mathbb{R}} |f| < \infty \}$

For $f \in L^1_{bc}(\mathbb{R})$, recall $\|f\|_1 = \int_{\mathbb{R}} |f|$ is the L^1-norm.

It's really a norm:

- $\|\lambda f\|_1 = |\lambda| \, \|f\|_1$,
- $\|f\|_1 \geq 0$ with equality iff $f = 0$,
- $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

Note In particular, $L^1_{bc}(\mathbb{R})$ is a $\mathbb{C}$-vector space.

Defineorem Let $f, g \in L^1_{bc}(\mathbb{R})$. Then the function

$$f * g : \mathbb{R} \longrightarrow \mathbb{C}$$
$$x \longmapsto \int_{\mathbb{R}} f(t)\, g(x-t)\, dt$$

is well-defined and called the convolution of f, g, and $f*g \in L^1_{bc}(\mathbb{R})$

Additionally, for $f, g, h \in L^1_{bc}(\mathbb{R})$,

(1) $f * g = g * f$

(2) $f * (g * h) = (f * g) * h$

(3) $f * (g + h) = f * g + f * h$.

$\lambda(f * g)$

$= (\lambda f) * g$

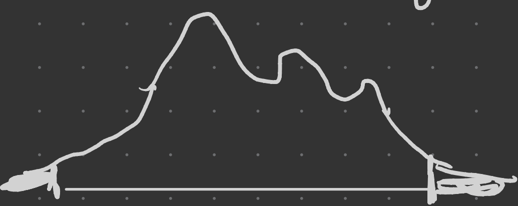
Pf Assume $|g(x)| \le C \ \forall x \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} |f(t)\, g(x-t)|\, dt \le C \int_{\mathbb{R}} |f(t)|\, dt = C \|f\|_1$$

so $f * g$ is well-defined and bounded. Now check cts:

Fix $x_0 \in \mathbb{R}$, assume $|f(x)|, |g(x)| \le C \ \forall x$, $g \neq 0$.

For a given $\varepsilon > 0$, $\exists T > |x_0|$ such that



$$\int_{|t|>T} |f(t)|\, dt < \frac{\varepsilon}{4C}$$

(why?) ✓

Claim $\exists \delta > 0$ s.t. $|x| \le 2T,\ |x - x'| < \delta \Rightarrow |g(x) - g(x')| < \dfrac{\varepsilon}{2\|g\|_1}$

(why?) ⟶ unif cts on compact $[-2T, 2T]$. ✓

Then for $|x-x_0|<\delta$,

$$\left| \int_{-T}^{T} f(t)g(x-t)\,dt - \int_{-T}^{T} f(t)g(x_0-t)\,dt \right|$$

$$\leq \int_{-T}^{T} |f(t)||g(x-t)-g(x_0-t)|\,dt$$

$$\leq \frac{\varepsilon}{2\|f\|_1}\int_{-T}^{T}|f(t)|\,dt \leq \frac{\varepsilon}{2},$$

and $$\int_{|t|>T} |f(t)||g(x-t)-g(x_0-t)|\,dt \leq 2C\int_{|t|>T}|f(t)|\,dt < \frac{\varepsilon}{2}.$$

Taken together, these imply $|x - x_0| < \delta \implies$

$$|f*g(x) - f*g(x_0)| < \varepsilon,$$

so $f*g$ is cts at x_0.

Now check $\|f*g\|_1 < \infty$:

$$\|f*g\|_1 = \int_{\mathbb{R}} |f*g| = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t)\, g(x-t)\, dt \right| dx$$

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t)\, g(x-t)|\, dt\, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t)\, g(x-t)|\, dx\, dt \qquad \text{[Fubini]}$$

$$= \int_{\mathbb{R}} |f(t)|\,dt \cdot \int_{\mathbb{R}} |g(x)|\,dx \qquad \text{(why?)} \checkmark$$

$$= \|f\|_1 \, \|g\|_1 \, .$$

Commutativity, associativity, distributivity — exc. $\square$

Defn For $f \in L^1_{bc}(\mathbb{R})$, the Fourier transform of f is

$$\hat{f} : \mathbb{R} \longrightarrow \mathbb{C}$$

$$\gamma \longmapsto \hat{f}(\gamma) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i \gamma x}\, dx \, .$$

Note $|\hat{f}(\gamma)| \leq \int_{\mathbb{R}} |f(x) e^{-2\pi i \gamma x}|\, dx = \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1$

so $\hat{f}$ is well-defined and bounded for $f \in L^1(\mathbb{R})$.