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Eigenbasis Method

L operator on $L^2(S^1)$ or $L^2([a,b])$ (x coord)

T operator on $L^2(\mathbb{R}_{>0})$ or $L^2(\mathbb{R}_{\geq 0})$ (t coord)

Goal: Find $u : S^1$ / or $[a,b]$ $\times \mathbb{R}_{>0} \longrightarrow \mathbb{C}$ satisfying $L(u) = T(u)$ and boundary, initial conditions.

Separation of variables seeks a solution of the form

$$u(x,t) = \sum \phi_n(x)\,\psi_n(t)$$

(1) Boundary conditions on x determine $\mathcal{D}(L)$. Show L Hermitian. $\Rightarrow$ eigenvalues real

(2) Find an eigenbasis for L: orthogonal basis (ϕ_n) of $\mathcal{H} = L^2(S^1)$ with $\phi_n \in \mathcal{D}(L)$, $L\phi_n = \lambda_n \phi_n$ for some $\lambda_n \in \mathbb{R}$.

(3) For each n solve the ODE $T\psi_n = \lambda_n \psi_n$ for ψ_n.

(4) Know L is diagonalizable wrt (ϕ_n); hope T is as well wrt ψ_n.

Then $u(x,t) = \sum \phi_n(x)\psi_n(t)$ is a formal solution to $Lu = Tu$. Remains to check initial condition, convergence, and validity of $T\sum(\cdots) = \sum T(\cdots)$ in (4).

$A \in F^{n\times n}$ diag'le then $\exists$ basis $v_1, \ldots, v_n$ of F^n s.t.
(1) $UAU^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \Leftrightarrow$ (2) $Av_i = \lambda_i v_i$

Tools for uniform convergence

Recall a sequence of fns $f_n : X \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ converges uniformly to $f : X \longrightarrow \mathbb{C}$ when $\forall \varepsilon > 0\ \exists N$ s.t. $\forall z \in X,\ n > N,\ |f_n(z) - f(z)| < \varepsilon$.

ind of z!

Lemma $(f_n : X \to \mathbb{C})$ converges uniformly to $f : X \to \mathbb{C}$ iff

$$\lim_{n \to \infty} \|f_n - f\|_\infty = 0.$$ I.e. uniform convergence = L^∞ convergence

for $\|g\|_\infty = \sup_{z \in X} |g(z)|$.

Pf Let $d_n := \|f_n - f\|_\infty = \sup_{z \in X} |f_n(z) - f(z)|$.

($\Leftarrow$) Suppose $\lim_{n \to \infty} d_n = 0$. Since $|f_n(z) - f(z)| \leq d_n \ \forall z \in X$,

$f_n(z) \xrightarrow[n \to \infty]{} f(z)$ with rate independent of z ✓

($\Rightarrow$) Suppose $\forall \varepsilon > 0 \ \exists N$ s.t. $\forall z \in X$, $n > N$, we have $|f_n(z) - f(z)| < \frac{\varepsilon}{2}$.

Then $d_n \leq \frac{\varepsilon}{2} < \varepsilon$ so $d_n \to 0$. $\square$

Weierstrass M-test Suppose $\emptyset \neq X \subseteq \mathbb{C}$, $g_n : X \to \mathbb{C}$, $M_n \geq 0$ with $\sum M_n$ convergent, and $|g_n(z)| \leq M_n \ \forall z \in X$.

Then $\sum g_n(z)$ converges absolutely and uniformly to some $f : X \to \mathbb{C}$.

Pf Observe $\left|\sum_{n=k}^{m} g_n(z)\right| \leq \sum_{n=k}^{m} |g_n(z)| \leq \sum_{n=k}^{m} M_n = \left|\sum_{n=k}^{m} M_n\right|$.

Since $\sum M_n$ converges, its partial sums satisfy the Cauchy criterion, so the same holds for $\sum g_n(z)$, ind. of z.

I.e. $\sum g_n$ is uniformly Cauchy $\Longrightarrow$ uniformly convergent.

All this holds for $\sum |g_n|$ as well, so $\sum g_n$ is absolutely conv. too. $\square$

Limits vs. integrals and derivatives

Thm Let $(f_n : [a,b] \to \mathbb{C})$ be a sequence of Riemann integrable fns converging uniformly to $f : [a,b] \to \mathbb{C}$. Then

$$\int_a^b f(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Pf AW. $\square$

Limits must converge with derivatives too! Right? .... right?

E.g. Set $f_n(x) = |x|^{1+1/n} \xrightarrow[n\to\infty]{} |x|$ uniformly in x

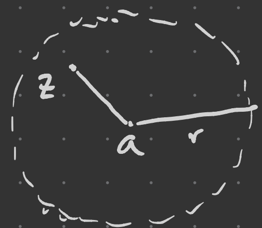
We have $f_n'(0) = 0$, but $|x|$ is not diff'l at 0.

Nonetheless,

Thm Let $\emptyset \neq X \subseteq \mathbb{C}$ open, $f_n : X \to \mathbb{C}$ diff'l converging pointwise to $f : X \to \mathbb{C}$. Suppose each f_n' cts and the sequence f_n' converges uniformly to some $g : X \to \mathbb{C}$. Then f is diff'l and $f'(z) = g(z)$.

Cor If $\sum g_n'(z)$ converges uniformly, each g_n' is cts, and $\sum g_n(z)$ converges, then $\frac{d}{dz}\left(\sum g_n(z)\right) = \sum g_n'(z)$. **Pf** Exc $\square$

Pf Thm Fix $a \in X$. WLOG, assume $X = B_r(a)$ for some $r > 0$.



For fixed $z \in B_r(a)$, define $u_z : [0,1] \to \mathbb{C}$
$$t \mapsto tz + (1-t)a$$

Then $u_z'(t) = z - a$ and $|u_z(t) - a| \leq |z - a|$,

with $u_z(0) = a$, $u_z(1) = z$.

Compute $I = \lim_{n \to \infty} \int_0^1 f_n'(u_z(t))\, u_z'(t)\, dt$ in two ways.

By substitution, $I = \lim_{n \to \infty} \left(f_n(u_z(1)) - f_n(u_z(0)) \right)$

$$= f(z) - f(a). \quad ①$$

On the other hand, $f_n'(u_z(t))$ converges unif on $[0,1]$, so $I = \int_0^1 \left(\lim_{n\to\infty} f_n'(u_z(t)) u_z'(t) \right) dt$

$$= \int_0^1 g(u_z(t))(z-a)\, dt$$

$$= (z-a) \int_0^1 g(u_z(t))\, dt \quad ②$$

$\int_0^1 g(a)\, dt$

Since ① = ②, $\dfrac{f(z)-f(a)}{z-a} - g(a) = \left(\int_0^1 g(u_z(t))\,dt \right) - g(a)$

$$= \int_0^1 (g(u_z(t)) - g(a))\, dt$$

Thus $\left| \dfrac{f(z)-f(a)}{z-a} - g(a) \right| \le \int_0^1 |g(u_z(t)) - g(a)|\, dt \xrightarrow[z\to a]{} 0$

so $f'(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = g(a)$. $\square$

Back to PDEs ...

Wave equation on S^1 Given $f, g \in L^2(S^1)$, find $u : S^1 \times \mathbb{R}_{>0} \to \mathbb{C}$

s.t. (D) $u(-, t_0) \in C^2(S^1)$, $u(x_0, -) \in C^2(\mathbb{R}_{>0})$

(IV) $\lim_{t \to 0^+} u(x,t) = f(x)$, $\lim_{t \to 0^+} \frac{\partial u}{\partial t}(x,t) = g(x)$.

(PDE) $$-\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial t^2}.$$

Apply the eigenbasis method: $L = \frac{-\partial^2}{\partial x^2}$, $T = \frac{-\partial^2}{\partial t^2}$

Know L is Hermitian with eigenbasis $(e_n)_{n\in\mathbb{Z}}$ and associated eigenvalues $\lambda_n = 4\pi^2 n^2 = (2\pi n)^2 > 0$. Set $\kappa_n = |2\pi n|$.

Have $f = \sum \hat{f}(n) e_n$, $g = \sum \hat{g}(n) e_n$ in L^2

Our ODE is $T\psi = \lambda_n \psi$, i.e. $\psi'' = -\kappa_n^2 \psi$ with solutions

$$\psi_n(t) = c_0 \cos(\kappa_n t) + \frac{c_1}{\kappa_n} \sin(\kappa_n t) \qquad \text{for } n \neq 0.$$

where $c_0 = \psi_n(0) = \hat{f}(n)$, $c_1 = \psi_n'(0) = \hat{g}(n)$

i.e. $$\psi_n(t) = \hat{f}(n) \cos(\kappa_n t) + \frac{\hat{g}(n)}{\kappa_n} \sin(\kappa_n t)$$

Thus the wave equation has formal sol'n

$$u(x,t) = \hat{f}(0) + t\,\hat{g}(0) + \sum_{n \neq 0} e_n(x)\,\psi_n(t)$$

See 11.3 Hsu for proving this converges, etc.