2025. II. 21

Recall the heat equation on S^1:

Given $f \in L^2(S^1)$, find $u : S^1 \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ such that

(D) For fixed $t_0 > 0$, $u(-, t_0) \in C^2(S^1)$, and for fixed $x_0 \in S^1$, $u(x_0, -) \in C^1(\mathbb{R}_{>0})$

(IV) $\forall x \in S^1$, $\lim_{t \to 0^+} u(x,t) = f(x)$

(PDE) $-\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial u}{\partial t}$

Potential solution

$$u(x,t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}.$$

Remains to justify:

regularity: $\sum \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}$ converges to

C^∞ fn of x, C^∞ fn of t.

IV $\lim_{t \to 0^+} \| u(-,t) - f \|_{L^2} = 0$.

or L^∞ for $f \in C^1(S^1)$

uniqueness our sol'n is the only one satisfying

(D), (IV), (PDE).

To prove regularity, we need to understand how decay rate of Fourier coefficients impacts differentiability.

Thm Fix $j \geq 1$, $f \in L^2(S^1)$. Suppose $\exists\, K \in \mathbb{R}$, $p > j+1$ s.t.

$\forall n \in \mathbb{Z} \setminus \{0\}$, $|\hat{f}(n)| \leq \frac{K}{|n|^p}$. Then the Fourier series of f

converges absolutely and uniformly to some $g \in C^j(S^1)$

such that
$$g^{(j)}(x) = \sum_{n \in \mathbb{Z}} (2\pi i n)^j \hat{f}(n) e_n(x)$$

and $f = g$ a.e.

Pf idea The Fourier coefficients of $\sum_{n \in \mathbb{Z}} (2\pi i n)^j \hat{f}(n) e_n(x)$

satisfy $\sum |(2\pi i n)^j \hat{f}(n)|^2 = \sum |2\pi n|^{2j} |\hat{f}(n)|^2 \leq \sum (2\pi)^{2j} K^2 \frac{1}{|n|^{2p-2j}}$

Now $2p-2j \geq 2$ so the series converges. $\square$

Cor Let $f \in L^2(S^1)$. Suppose $\forall p > 2$ $\exists K_p \in \mathbb{R}$ s.t. $\forall n \in \mathbb{Z} \setminus \{0\}$,

$|\hat{f}(n)| \leq \dfrac{K_p}{|n|^p}$. Then the Fourier series of f converges uniformly to some $g \in C^\infty(S^1)$ with $g = f$ a.e. $\square$

Now back to regularity: $\sum \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}$ converges to C^∞ fn of x, C^∞ fn of t.

For convenience, write $u(x,t)$ for this series. Then for fixed $t_0 > 0$

$$\widehat{u(-,t_0)}(n) = \hat{f}(n) e^{-4\pi^2 n^2 t_0}$$

Since $f \in L^2(S^1)$, $\lim_{n \to \pm\infty} \hat{f}(n) = 0$, so in particular $\exists K > 0$ s.t.

$|\hat{f}(n)| \leq K$ $\forall n \in \mathbb{N}$.

Thus $\left|\hat{f}(n) e^{-4\pi^2 n^2 t_0}\right| \leq K\left(e^{4\pi^2 t_0}\right)^{-n^2} \ll \frac{1}{|n|^p} \quad \forall p.$

By the corollary above, $u(-,t_0) \in C^\infty(S^1)$.

Smoothness in the other variable is similar. $\square$

__Note__ Even though $u(-,0) = f$ need not be smooth, or even cts, $u(-,t_0)$ is smooth $\forall t_0 > 0$.

Now on to __IV__ $\lim_{t \to 0^+} \| u(x,t) - f(x) \|_{L^2} = 0$.

or L^∞ for $f \in C^1(S^1)$

__Pf__ $u(x,t) - f(x) = \sum \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t} - \sum \hat{f}(n) e_n(x)$

$= \sum \left(e^{-4\pi^2 n^2 t} - 1\right) \hat{f}(n) e_n(x).$

If $g(t) = \|u(-,t) - f\|^2$, then by the isometry theorem,

$$g(t) = \sum_{n \in \mathbb{Z}} |1 - e^{-4\pi^2 n^2 t}|^2 \, |\hat{f}(n)|^2$$

For $t \geq 0$, $0 < e^{-4\pi^2 n^2 t} \leq 1$ so $\leq |\hat{f}(n)|^2$.

By the M-test with $M_n = |\hat{f}(n)|^2$, $g(t)$ converges uniformly to a continuous function on $\mathbb{R}_{\geq 0}$. By continuity,

$$\lim_{t \to 0^+} \|u(-,t) - f\|^2 = g(0) = \sum |1 - e^0|^2 \, |\hat{f}(n)|^2 = 0. \quad \square$$

Thm If additionally $f \in C^1(S^1)$, then $\lim_{t \to 0^+} \|u(-,t) - f\|_\infty = 0$.

Recall $\|g\|_\infty = \sup \{ |g(x)| \}$, so this is pointwise convergence.

Thm [M-test]

$X \subseteq \mathbb{C}$, $\quad f_n : X \longrightarrow \mathbb{C}$, $\quad M_n \geq 0$

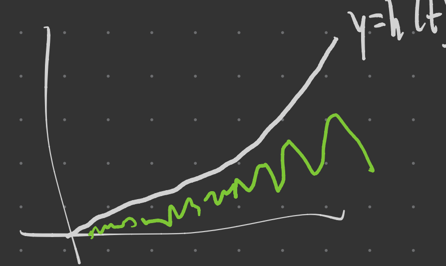
such that $\forall x \in X$, $|f_n(x)| \leq M_n$ and $\sum M_n < \infty$.

Then (f_n) converges uniformly to some $f : X \longrightarrow \mathbb{C}$.

Pf The assignment $\eta(t) = \| u(-,t) - f \|_\infty$ is a nonnegative function.

By the squeeze theorem, it suffices to construct $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$

cts s.t. $\eta(t) \leq h(t)$ with $h(0) = 0$.



Set $h(t) = \sum_{n \in \mathbb{Z}} |1 - e^{-4\pi^2 n^2 t}| |\hat{f}(n)|$.

We have

$$\eta(t) = \sup_{x \in S^1} |u(x,t) - f(x)|$$

$$= \sup_{x \in S^1} \left| \sum_n \left(e^{-4\pi^2 n^2 t} - 1\right) \hat{f}(n) e_n(x) \right|$$

$$\leq \sup_{x \in S^1} \sum_n \left| \left(e^{-4\pi^2 n^2 t} - 1\right) \hat{f}(n) e_n(x) \right|$$

$$= \sup_{x \in S^1} \sum_n \left| e^{-4\pi^2 n^2 t} - 1 \right| \left| \hat{f}(n) \right| \left| e_n(x) \right| \qquad 1$$

$$= \sum_n \left| e^{-4\pi^2 n^2 t} - 1 \right| \left| \hat{f}(n) \right|$$

$$= h(t).$$

M $\Rightarrow$ unif conv of cts summands $\Rightarrow$ cts. $\square$

Finally, uniqueness: Suppose $f \in C^0(S^1)$, $u : S^1 \times \mathbb{R}_{\geq 0} \to \mathbb{C}$ s.t.

(D) $u(-,t) \in C^2(S^1)$ $\forall t > 0$, $u(x,-) \in C^1(\mathbb{R}_{\geq 0})$

(IV) u is cts and $u(-,0) = f$

(PDE) $\forall t > 0$, $\Delta u = -\dfrac{\partial u}{\partial t}$.

Then $u(x,t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}$.

To prove this, we need the following:

Thm Suppose $f \in C^1(S^1)$. Then the Fourier series of f converges absolutely and uniformly to f.

Extra Derivative lemma If $g \in L^2(S^1)$, then $\sum_{n \in \mathbb{Z} \setminus 0} \frac{1}{2\pi n} \hat{g}(n)$ converges absolutely.

Pf lemma Observe that $(a_n)_{n \in \mathbb{Z}}$, $a_n = \begin{cases} 1 & n = 0 \\ \frac{1}{|2\pi n|} & n \neq 0 \end{cases}$ is in $\ell^2(\mathbb{Z})$.

Since $g \in L^2(S^1)$ has L^2-convergent Fourier series, $(|\hat{g}(n)|) \in \ell^2(\mathbb{Z})$.

Thus $\langle (|\hat{g}(n)|), (a_n) \rangle = |\hat{g}(0)| + \sum_{n \in \mathbb{Z} \setminus 0} \left| \frac{1}{2\pi n} \hat{g}(n) \right| < \infty$. $\square$

Pf Thm We have $f' \in C^0(S^1)$ and $\widehat{f'}(n) = 2\pi i n \hat{f}(n)$.

By the extra derivative lemma,

$$\sum_{n \neq 0} \frac{1}{|2\pi n|} |2\pi i n| |\hat{f}(n)| = \sum_{n \neq 0} |\hat{f}(n)| < \infty.$$

Set $g(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e_n(x)$. By an M-test with $M_n = |\hat{f}(n)|$,

$g(x)$ converges uniformly to a fn in $C^0(S^1)$.

Since $\hat{g}(n) = \hat{f}(n)$, $f = g$ a.e. Since f, g both cts,

we get $f = g$ on the nose. $\square$

Pf of uniqueness Since $u(-, t_0) \in C^2(S^1)$ for $t_0 > 0$,

$u(-, t_0)$ converges uniformly to its Fourier series.

Set $\varphi_n(t) = \widehat{u(-, t)}(n) = \int_0^1 u(x,t)\, e_{-n}(x)\, dx$.

Since u is $C^2 \supseteq C^1$, we can differentiate inside the integral:

$$\psi_n'(t) = \int_0^1 \frac{\partial u}{\partial t} e_{-n}(x)\,dx$$

$$= \int_0^1 \frac{\partial^2 u}{\partial x^2} e_{-n}(x)\,dx$$

$$= (2\pi i n)^2 \psi_n(t)$$

$$= -4\pi^2 n^2 \psi_n(t).$$

Since $\psi_n(0) = \hat{f}(n)$, we get $\psi_n(t) = \hat{f}(n) e^{-4\pi^2 n^2 t}$. $\square$