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Partial differential equations

Heat

- Wire with uniform density ρ.
- $u(x,t)$ = temp of wire at pos'n x time t

Q Given initial condition $u(x,0) = f(x)$, what is $u(x,t)$?

A Will be specified by a differential equation (+ boundary conditions)

Defn $q(x,t)$ = rate of flow of heat energy per unit length at (x,t)

Fourier's law of heat conduction $\quad q = -k \dfrac{\partial u}{\partial x}$

k: thermal conductivity

Defn $Q(x,t)$ = heat density, so

$$\frac{\partial Q}{\partial t} = c\rho \frac{\partial u}{\partial t}$$

specific heat capacity of medium

By conservation of energy,

$$\frac{\partial Q}{\partial t} = -\frac{\partial q}{\partial x}$$

So $$\frac{\partial u}{\partial t} = \frac{-1}{c\rho}\frac{\partial q}{\partial x} = \frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2}$$

This gives us the <u>heat equation</u> $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

(in units with $\frac{k}{c\rho} = 1$).

Full specification for heat eq'n on circle:

Given $f \in L^2(S^1)$, find $u : S^1 \times \mathbb{R}_{>0} \to \mathbb{R}$ such that

(D) For fixed $t_0 > 0$, $u(-, t_0) \in C^2(S^1)$, and for fixed $x_0 \in S^1$, $u(x_0, -) \in C^1(\mathbb{R}_{>0})$

(IV) $\forall x \in S^1$, $\lim_{t \to 0^+} u(x,t) = f(x)$

(PDE) $-\frac{\partial^2 u}{\partial x^2} = -\frac{\partial u}{\partial t}$

<u>Note</u> Disc^ts initial condition is OK!

Note (PDE) is equiv to $\Delta u = \frac{-\partial u}{\partial t}$.

($\Delta = \nabla^2$ notation)

Laplacian (positive operator)

Recall $(e_n)_{n \in \mathbb{Z}}$ is an eigenbasis for Δ

Since $f \in L^2(S^1)$, $f(x) = \sum \hat{f}(n) e_n(x)$ in L^2.

Let $\psi_n(t) = \widehat{u(-,t)}(n)$. Since $u(-,t) \in C^2(S^1) \subseteq L^2(S^1)$

have $u(x,t) = \sum \psi_n(t) e_n(x)$ in L^2.

Know Δ diagonalizes

Hope: $-\frac{\partial}{\partial t}$ diagonalizes it as well.

If so,

$$\Delta(u)(x,t) = \sum 4\pi^2 n^2 e_n(x)\, \psi_n(t)$$

$$-\frac{\partial u}{\partial t}(x,t) = \sum (-1)\, e_n(x)\, \psi_n'(t)$$

$\frac{\partial}{\partial t}$ commutes with $\sum_{n\in\mathbb{Z}}$

Thus we seek functions $\psi_n(t)$ cts at 0 such that

$$-\psi_n'(t) = 4\pi^2 n^2\, \psi_n(t) \quad \text{and} \quad \psi_n(0) = \hat{f}(n).$$

$$\frac{d}{dt} e^{at} = a e^{at}$$

$\lim_{t\to 0^+}$ commutes with $\sum_{n\in\mathbb{Z}}$?

Q Solution for $\psi_n(t)$?

A $\psi_n(t) = \hat{f}(n)\, e^{-4\pi^2 n^2 t}$

Thus $\boxed{u(x,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}}$ is our candidate solution.

Now need to formally justify:

regularity: $\sum \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}$ converges to C^∞ fn of x, C^∞ fn of t.

IV $\lim_{t\to 0^+} \| u(x,t) - f(x) \|_{L^2} = 0$.

or L^∞ for $f \in C^1(S^1)$

uniqueness our sol'n is the only one satisfying (D), (IV), (PDE).