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Operators on Hilbert spaces

In preparation for our treatment of PDE's via the eigenbasis method, we develop the language of (Hermitian) operators on a Hilbert space.

Defn Let $\mathcal{H}$ be a Hilbert space. An operator on $\mathcal{H}$ is a linear map $T : \mathcal{D}(T) \longrightarrow \mathcal{H}$ where $\mathcal{D}(T) \leq \mathcal{H}$.

E.g. $X = S^1$ or $[a,b]$, $\mathcal{H} = L^2(X)$. Then
$$D : C^1(X) \longrightarrow \mathcal{H}, \quad f \longmapsto -if'$$
is an operator on $\mathcal{H}$.

Note $D(f) = \sum_{n \in \mathbb{Z}} 2\pi n \hat{f}(n) e_n$

E.g. For $\mathcal{H}$ a separable Hilbert space with orthonormal basis $\mathcal{B} = (e_n)_{n \geq 1}$, define

$$\mathcal{H}_0 := \left\{ \sum_{n=1}^{\infty} c_n e_n \;\middle|\; \text{all but finitely many } c_n = 0 \right\}.$$

Define operators μ, ι on $\mathcal{H}$ by formulae

$$\mu\left(\sum_{n=1}^{\infty} c_n e_n\right) = \sum_{n=1}^{\infty} n c_n e_n$$

$$\iota\left(\sum_{n=1}^{\infty} c_n e_n\right) = \sum_{n=1}^{\infty} \left(\frac{c_n}{n}\right) e_n$$

with $\mathcal{D}(\mu) = \mathcal{H}_0$, $\mathcal{D}(\iota) = \mathcal{H}$.

E.g. More generally, for $\alpha : \mathbb{Z}_{\geq 1} \to \mathbb{C}$ any function,

$$\alpha\left(\sum_{n=1}^{\infty} c_n e_n\right) = \sum_{n=1}^{\infty} \alpha(n) c_n e_n$$

defines a linear operator on $\mathcal{H}$ with $\mathcal{D}(\alpha) = \mathcal{H}_0$.

Defn Such an operator is called a *diagonal operator* with respect to (e_n).

E.g. The operator $D : f \longmapsto -if'$ on $L^2(S^1)$ is diagonalized by the standard basis $(e_n)_{n \in \mathbb{Z}}$.

Defn An operator T on $\mathcal{H}$ is bounded when $\exists M > 0$ s.t. $\forall f \in \mathcal{H}$, $\|T(f)\| \leq M\|f\|$.

Thm TFAE: (UC) T is uniformly continuous

(CO) T is continuous at $0 \in \mathcal{D}(T)$.

(B) T is bounded.

Pf HW. $\square$

Hermitian and positive operators

Defn An operator T on $\mathcal{H}$ is Hermitian (or self-adjoint) when $\forall f, g \in \mathcal{D}(T)$, $\langle T(f), g \rangle = \langle f, T(g) \rangle$.

Remark Bounded operators on a Hilbert space always admit an adjoint T^* satisfying $\langle T(f), g\rangle = \langle f, T^* g\rangle$.
(This is a consequence of the "Riesz representation theorem".)

Hermitian operators satisfy $T = T^*$.

operator : Hermitian operator :: complex #s : real #s

E.g. Consider $D : f \mapsto -if'$ on $L^2(S^1)$ again, with $\mathcal{D}(D) = C^1(S^1)$.

For $f, g \in C^1(S^1)$,

$$\langle D(f), g\rangle = \int_0^1 \underbrace{(-i)f'(x)}_{dv}\, \underbrace{\overline{g(x)}}_{u}\, \underline{dx}$$

$$= (-i)f(x)\overline{g(x)}\Big|_0^1 - \int_0^1 (-i)f(x)\overline{g'(x)}\,dx$$

$$= (-i)\cancelto{0}{\left(f(1)\overline{g(1)} - f(0)\overline{g(0)}\right)} + \int_0^1 i f(x)\overline{g'(x)}\,dx$$

$$= \int_0^1 f(x)\overline{\left((-i)g'(x)\right)}\,dx$$

$$= \langle f, D(g)\rangle .$$

Thus D is Hermitian.

<u>Q</u> For which $\alpha : \mathbb{N} \to \mathbb{C}$ is $\alpha\left(\sum c_n e_n\right) = \sum \alpha(n) c_n e_n$ Hermitian? ($\mathcal{H}$ Hibert space with orthonormal basis (e_n).)

<u>A</u> $\langle \alpha(\sum c_n e_n), \sum d_n e_n \rangle = \langle \sum \alpha(n) c_n e_n, \sum d_n e_n \rangle \quad \forall n.$

$$= \sum \alpha(n) c_n \overline{d_n} \overset{?}{=} \sum c_n \overline{\alpha(n) d_n} \quad \text{iff } \alpha(n) \in \mathbb{R}$$

E.g. The shift $\sigma : \sum c_n e_n \longmapsto \sum c_n e_{n+1}$ is a non-Hermitian operator. (moral exc — not Hermitian)

Thm Let T be a Hermitian operator. Then $\forall f \in \mathcal{D}(T)$, $\langle T(f), f \rangle$ is a real number.

Pf By conjugate symmetry, $\langle f, T(f) \rangle = \overline{\langle T(f), f \rangle}$, but the LHS also equals $\langle T(f), f \rangle$ by self-adjointness. Thus $\langle T(f), f \rangle = \overline{\langle T(f), f \rangle}$ is real. $\square$

Defn A Hermitian operator is positive when $\langle T(f), f \rangle \geq 0$ for all f.

Can have $\langle Tf, f \rangle = 0$ for $f \neq 0$.

E.g. On $L^2(S^1)$, $\Delta : f \longmapsto -f''$ with $\mathcal{D}(\Delta) = C^2(S^1)$ is called the Laplacian operator. Observe:

$$\langle \Delta(f), g \rangle = -\int_0^1 \underbrace{f''(x)}_{dv}\, \underbrace{\overline{g(x)}}_{u}\, dx$$

$$= \underbrace{-f'(x)\overline{g(x)}\Big|_0^1}_{0} + \int_0^1 f'(x)\overline{g'(x)}\, dx$$

$$= \langle f', g' \rangle.$$

Similarly, $\langle f, \Delta(g) \rangle = \langle f', g' \rangle$ so Δ is Hermitian.

Moreover, $\langle \Delta(f), f \rangle = \|f'\|^2 \geq 0$ so Δ is positive.

E.g. A diagonal operator associated with $\alpha : \mathbb{N} \to \mathbb{C}$ is positive iff $\alpha(n) \geq 0 \ \forall n$.

Thm A Hermitian operator's eigenvalues are all real. If the operator is positive, then the eigenvalues are also nonnegative.

Pf Suppose $T : \mathcal{D}(T) \to \mathcal{H}$ is Hermitian with $T(f) = \lambda f$ for some $f \in \mathcal{D}(T) \setminus 0$. Then $\mathbb{R} \ni \langle T(f), f \rangle = \langle \lambda f, f \rangle = \lambda \|f\|^2$

$\Rightarrow \lambda \in \mathbb{R}$. If T is positive, then $\langle T(f), f \rangle \geq 0$ so $\lambda \geq 0$. $\square$

Thm Let T be a Hermitian operator, $\{u_1, \dots, u_n\}$ a set of eigenvectors of T with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\{u_1, \dots, u_n\}$ is an orthogonal set.

Pf WTS: $\langle u_i, u_j \rangle = 0$ for $i \neq j$.

Know: $\langle Tu_i, u_j \rangle = \langle u_i, Tu_j \rangle$ [Hermitian]

$$\begin{array}{ccc} \langle Tu_i, u_j \rangle & = & \langle u_i, Tu_j \rangle \\ \text{[eigen]}\,\| & & \| \\ \langle \lambda_i u_i, u_j \rangle & & \langle u_i, \lambda_j u_j \rangle \\ \| & & \| \\ \lambda_i \langle u_i, u_j \rangle & & \overline{\lambda_j} \langle u_i, u_j \rangle \end{array}$$

so if $\langle u_i, u_j \rangle \neq 0 \Rightarrow \lambda_i = \overline{\lambda_j} = \lambda_j$

eigenval of Hermitian are real. ☺ □

E.g. Let $\mathcal{H} = L^2(S^1)$, $D(f) = -if'$, $\Delta(f) = -f''$.

Claim $(e_n \mid n \in \mathbb{Z})$ is an eigenbasis of D and of Δ with eigenvalues $2\pi n$ and $4\pi^2 n^2$, resp.

Check $D(e_n) = (-i) 2\pi i n \, e^{2\pi i n x}$

$= 2\pi n \, e_n$ ✓

$\Delta(e_n) = D(D(e_n)) = 4\pi^2 n^2 e_n$ ✓