2025. II. 12

Riemann ζ-values

Defn For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define
$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s};$$
this is the Riemann ζ-function.

Notes
- Basic tools from complex analysis provide a unique meromorphic continuation of ζ to $\mathbb{C}$ (with simple pole at $s=1$).
- Riemann hypothesis: If $\zeta(s) = 0$ and $0 < \mathrm{Re}(s) < 1$, then $\mathrm{Re}(s) = \frac{1}{2}$.

Thm [Euler, 1737] For $\mathrm{Re}(s) > 1$,
$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Pf Each term $\frac{1}{1-p^{-s}}$ can be expanded as a geometric series

$1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots$, which converges absolutely for $\mathrm{Re}(s)>1$.

Note that
$$\frac{1}{1-p^{-s}}\cdot\frac{1}{1-q^{-s}}=\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\left(1+\frac{1}{q^s}+\frac{1}{q^{2s}}+\cdots\right)$$
$$=1+\frac{1}{p^s}+\frac{1}{q^s}+\frac{1}{(p^2)^s}+\frac{1}{(pq)^s}+\frac{1}{(q^2)^s}+\cdots$$
$$=\sum_{n=p^aq^b}\frac{1}{n^s}.$$

Proceed by induction on max size of prime factors to get
$$\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}=\zeta(s). \quad \square$$

Cor There are infinitely many primes.

Pf $\infty = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \zeta(1) \stackrel{?}{=} \prod \frac{1}{1 - 1/p} = \prod \frac{p}{p-1}$. $\square$

$\lim_{N\to\infty} \Pr(s$ int's from $\{1, \ldots, N\}$ being coprime$)$

Cor The asymptotic probability that s randomly selected positive integers share no common factors >1 is $\frac{1}{\zeta(s)}$.

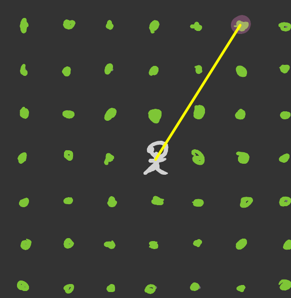
Pf We have $\frac{1}{\zeta(s)} = \left(\prod_p \frac{1}{1-p^{-s}}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\right)$. Now

$\frac{1}{p^s} = \text{Prob}(s$ pos integers all divisible by $p)$ (Why?)

so $1 - \frac{1}{p^s} = \text{Prob}($at least one of s positive integers not div by $p)$.

By independence, $\frac{1}{\zeta(s)} = \text{Prob}(s$ pos integers share no prime factor$)$. $\square$

Cor Suppose trees are planted in a square grid and you are standing at $(0,0)$. Pick a tree at random, the probability that your view of the tree is not obstructed by another tree is $\frac{1}{\zeta(2)}$. $\square$



Goal Compute $\zeta(s)$ for $s \in \mathbb{Z}_{\geq 2}$.

We will fail! We'll get $\zeta(2s)$, $s \geq 1$ integer

$\zeta(2s+1)$ — unknown!

Bernoulli polynomials

Today's convention: define functions on $[0,1)$ then extend periodically

Inductive Defn $B_1(x) = x - \frac{1}{2}$

$$\frac{d}{dx} B_k(x) = B_{k-1}(x)$$

$$\int_0^1 B_k(x)\, dx = 0$$

Q1 Are the $B_k(x)$ well-defined polynomials?

Q2 Determine $B_2(x)$.

let's compute $\hat{B}_k(n) = \int_0^1 \overbrace{B_k(x)}^{u} \overbrace{e^{-2\pi i n x}\,dx}^{dv}$

$$= B_k(x) \frac{e^{-2\pi i n x}}{-2\pi i n}\Bigg|_0^1 - \int_0^1 \frac{e^{-2\pi i n x}}{-2\pi i n} B_{k-1}(x)\,dx$$

$$= \frac{1}{-2\pi i n} B_k(x)\Big|_0^1 + \frac{1}{2\pi i n}\hat{B}_{k-1}(n)$$

$$= \frac{1}{-2\pi i n}\int_0^1 B_k'(x)\,dx + \frac{1}{2\pi i n}\hat{B}_{k-1}(n)$$

$$= \frac{1}{2\pi i n}\hat{B}_{k-1}(n).$$

Since $\hat{B}_1(n) = \int_0^1 (x-\frac{1}{2}) e^{-2\pi i n x} dx = \frac{-1}{2\pi i n}$ (comp'n)

we learn that

$$\hat{B}_k(n) = \frac{-1}{(2\pi i n)^k}.$$

Thus the Fourier series of $B_k(x)$ is

$$\frac{-1}{(2\pi i)^k} \sum_{0 \neq n \in \mathbb{Z}} \frac{e^{2\pi i n x}}{n^k}.$$

Claim For $k>1$, $B_k(0) = B_k(1) \implies B_k$ cts on S^1 and the Fourier series converges pointwise.

Pf $B_k(1) - B_k(0) = \int_0^1 B_{k-1}(x)\,dx = 0$ for $k>1$ + IOU ptwise convergence of Fourier series $\square$ of cts

Thm $B_{2s}(0) = \dfrac{-2\zeta(2s)}{(2\pi i)^{2s}}$ for $s \geq 1$ and

$B_{2s+1}(0) = 0$ for $2s+1 > 1$. $\rightsquigarrow$ cancelling $\pm n$ terms

In particular,

$$\boxed{\zeta(2s) = (-1)^{s+1} 2^{2s-1} \pi^{2s} B_{2s}(0).}$$

Prop The polynomials $B_k(x)$ have generating function

$$1 + tB_1(x) + t^2 B_2(x) + t^3 B_3(x) + \cdots = \frac{te^{tx}}{e^t - 1} \in \mathbb{C}[\![t, x]\!]$$

Pf Define $f(t,x) = \text{LHS}$. Then

$$\frac{\partial}{\partial x} f(t,x) = t \cdot 1 + t^2 B_1(x) + t^3 B_2(x) + \cdots = t \cdot f(t,x).$$

Thus $f(t,x) = C(t) \cdot e^{tx}$ for some $C(t) \in \mathbb{C}[\![t]\!]$.

To compute $C(t)$, observe

$$\int_0^1 f(t,x)\,dx = \int_0^1 1 \cdot dt + \sum_{l \geq 1} t^l \int_0^1 B_l(x)\,dx$$

$$= 1$$

while $\int_0^1 C(t) e^{tx}\,dx = C(t) \int_0^1 e^{tx}\,dx = C(t) \left.\frac{e^{tx}}{t}\right|_{x=0}^{x=1} = C(t) \frac{e^t - 1}{t}$.

Thus $C(t) = \dfrac{t}{e^t - 1}$ and $f(t,x) = \dfrac{te^{tx}}{e^t - 1}$. $\square$

Evaluating at $x=0$ gives

$$1 + t B_1(0) + t^2 B_2(0) + t^3 B_3(0) + \cdots = \frac{t}{e^t - 1}$$

$$\Rightarrow \underbrace{1 - \frac{t}{2} - \frac{t}{e^t - 1}}_{} = t^2 B_2(0) + t^4 B_4(0) + t^6 B_6(0) + \cdots$$

Taylor expand: $\frac{-t^2}{12} + \frac{t^4}{720} - \frac{t^6}{30240} + \frac{t^8}{1209600} - \cdots$

Hence $B_2(0) = \frac{-1}{12} \Longrightarrow \zeta(2) = \frac{\pi^2}{6}$

$$\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

$B_4(0) = \frac{1}{720} \Longrightarrow \zeta(4) = \frac{\pi^4}{90}$

$= \text{Prob}($ 2 pos ints being coprime $)$

$B_6(0) = \frac{-1}{30240} \Longrightarrow \zeta(6) = \frac{\pi^6}{945}$

$\approx 60.7\%$