2025.II.10

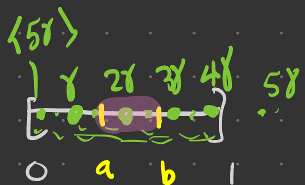
Weyl's equidistribution theorem

For $x \in \mathbb{R}$, let $\langle x \rangle$ denote the fractional part of x — i.e.

$0 \leq \langle x \rangle < 1$ and $x - \langle x \rangle \in \mathbb{Z}$.

Thm [Weyl] If γ is irrational, then for every $0 \leq a \leq b \leq 1$,

$$\frac{1}{n} \left| \left\{ r \in \{1, \dots, n\} \;\middle|\; a \leq \langle r\gamma \rangle \leq b \right\} \right| \xrightarrow[n \to \infty]{} b - a .$$



Note View $\langle x \rangle$ as a representative of $x + \mathbb{Z}$ in $\mathbb{R}/\mathbb{Z} = S^1$.

Equivalently, looking at $e^{2\pi i x}$

Thm 2 [Weyl] Suppose γ irrational. If $f: S^1 \to \mathbb{C}$ is continuous,

then $\frac{1}{n} \sum_{r=1}^{n} f(r\gamma) \xrightarrow[n\to\infty]{} \int_{S^1} f$.

Pf that Thm 2 $\Rightarrow$ Thm 1 Fix $\varepsilon > 0$. Construct $f_+, f_- : S^1 \to \mathbb{R}$

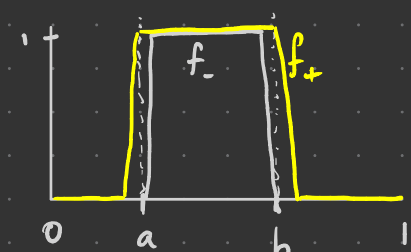
so that (a) $f_+(t) \geq 1 \geq f_-(t)$ for $t \in [a,b]$.

(b) $f_+ \geq 0$

(c) $f_-(t) = 0$ for $t \notin [a,b]$

(d) $(b-a) + \varepsilon \geq \int_{S^1} f_+$

(e) $\int_{S^1} f_- \geq (b-a) - \varepsilon$



Then $\sum_{r=1}^{n} f_+(r\gamma) \geq \left|\{r \in \{1,\dots,n\} \mid \langle r\gamma\rangle \in [a,b]\}\right| \geq \sum_{r=1}^{n} f_-(r\gamma)$.

$= \sum_{r=1}^{n} \chi_{[a,b]}(r\gamma)$

By Thm 2, we can take N s.t. for $n \geq N$,

$$\left| \frac{1}{n} \sum_{r=1}^{n} f_\pm(r\gamma) - \int_{S^1} f_\pm \right| \leq \varepsilon$$

and thus

$$\varepsilon + \int_{S^1} f_+ \geq \frac{1}{n} \left|\{r \in \{1,\dots,n\} \mid \langle r\gamma\rangle \in [a,b]\}\right| \geq \int_{S^1} f_- - \varepsilon$$

$$\Rightarrow \quad 2\varepsilon + (b-a) \geq \frac{1}{n} \left| \{r \in \{1,\dots,n\} \mid \langle r\gamma\rangle \in [a,b]\} \right| \geq (b-a] + 2\varepsilon.$$

Since ε was arbitrary, we conclude that

$$\frac{1}{n}\left|\left\{r\in\{1,\dots,n\} \mid \langle r\gamma\rangle \in [a,b]\right\}\right| \xrightarrow[n\to\infty]{} b-a. \quad \square$$

Pf of Thm 2 Write $G_n(f) := \frac{1}{n}\sum_{r=1}^{n} f(r\gamma) - \int_{S^1} f$.

We aim to show $G_n(f) \xrightarrow[n\to\infty]{} 0$.

Step 1 If $f=1$, then $G_n(1) = \frac{1}{n}\cdot n - \int_{S^1} 1 = 0$. ✓

Step 2 If $f=e_s$, $s\in\mathbb{Z}\setminus\{0\}$, then $\quad e_s(x) = e^{2\pi i s x}$

$$|G_n(e_s)| = \left|\frac{1}{n}\sum_{r=1}^{n} e^{2\pi i s r\gamma} - \int_{S^1} e^{2\pi i s x}\,dx\right|$$

$$= \left| \frac{1}{n} e^{2\pi i s \gamma} \sum_{r=0}^{n-1} e^{2\pi i s r \gamma} - 0 \right|$$

$$= \left| \frac{1}{n} e^{2\pi i s \gamma} \cdot \frac{e^{2\pi i s n \gamma} - 1}{e^{2\pi i s \gamma} - 1} \right|$$

$$= \frac{1}{n} \left| \frac{e^{2\pi i s n \gamma} - 1}{e^{2\pi i s \gamma} - 1} \right|$$

$$\leq \frac{1}{n} \cdot \underbrace{\frac{2}{|e^{2\pi i s \gamma} - 1|}}_{\text{does not depend on } n \text{ and } \neq 0 \text{ b/c } \gamma \in \mathbb{R} \setminus \mathbb{Q}} \xrightarrow[n \to \infty]{} 0. \checkmark$$

Step 3 If $f = \sum_{s=-m}^{m} c_s e_s$ is a trig polynomial then

$G_n(f) \longrightarrow 0$ as well by linearity of $\int_{S^1}$. $\checkmark$

<u>Step 4</u> If $f, g \in C^0(S^1)$ and $\|f-g\|_\infty \leq \varepsilon$

$\|f-g\|_\infty = \sup_{x \in S^1} |f(x)-g(x)|$

(i.e. $|f(t)-g(t)| \leq \varepsilon \ \forall t \in S^1$), then

$$|G_n(f) - G_n(g)| \leq \frac{1}{n}\sum_{r=1}^{n} |f(r\gamma)-g(r\gamma)| + \int_{S^1} |f-g|$$

$$\leq \varepsilon + \varepsilon = 2\varepsilon \quad \forall n \geq 0.$$

Césaro sum of f_N

<u>Step 5</u> If $f \in C^0(S^1)$ and $\varepsilon > 0$, then there is a trig poly P with $|P(t)-f(t)| \leq \frac{\varepsilon}{3}$ $\forall t \in S^1$. By Step 3, we can take N s.t. $|G_n(P)| \leq \frac{\varepsilon}{3}$ for $n \geq N$. By Step 4,

$|G_n(f) - G_n(P)| \leq \frac{2\varepsilon}{3}$ and so

$$|G_n(f)| \leq |G_n(P)| + |G_n(f) - G_n(P)| \leq \varepsilon$$

for all $n \geq N$. Hence $|G_n(f)| \longrightarrow 0$ as $n \to \infty$. $\square$

<u>Challenge</u> Adapt the above proof to show Weyl's equidistribution criterion: a sequence (a_r) equidistributes mod $\mathbb{Z}$ iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} e^{2\pi i s a_r} = 0 \quad \forall s \neq 0.$$

✓

<u>Challenge</u> Show that $\left(\left(\frac{1+\sqrt{5}}{2}\right)^r\right)_r$ does <u>not</u> equidistribute over $\mathbb{R}/\mathbb{Z}$.

Multiple dimns: If $\gamma_1, \dots, \gamma_k$ are irrational and

$\dim_{\mathbb{Q}} \operatorname{span}\{1, \gamma_1, \dots, \gamma_k\} = k+1$ then

$(\langle r\gamma_1 \rangle, \dots, \langle r\gamma_k \rangle)_r$ equidistributes over

$\mathbb{R}^k/\mathbb{Z}^k = (S^1)^k.$

For $S \subseteq \mathbb{R}^k/\mathbb{Z}^k$ measurable,

$$\lim_{N\to\infty} \frac{1}{N} \left| \{ r \in \{1, \dots, N\} \mid (\langle r\gamma_1 \rangle, \dots, \langle r\gamma_k \rangle) \in S \} \right|$$

$= \mu(S)$