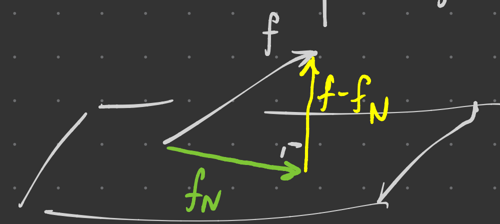
2025. II. 5

Bessel's inequality $\|f_N\| \le \|f\|$.

Pf Recall $f_N = \sum_{n=-N}^{N} \hat{f}(n) e_n = \mathrm{proj}_{F_N} f$ where

$F_N = \mathrm{span}\{e_{-N}, \dots, e_N\} \le L^2(S^1)$ is the subspace of degree N trigonometric polynomials.



We have $f - f_N \perp f_N$

so by Pythagoras

$$\|f\|^2 = \|f - f_N + f_N\|^2 = \underbrace{\|f - f_N\|^2}_{\geq 0} + \|f_N\|^2 \geq \|f_N\|^2 .$$

$\square$

Defn The Dirichlet kernel $\{D_N \mid N \in \mathbb{N}\}$ is

$$D_N(x) := \sum_{n=-N}^{N} e_n(x).$$

The Fejér kernel $\{F_N \mid N \geq 1\}$ is

$$F_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(x).$$

(see demo)

Thm For $f \in C^0(S^1)$, $f * D_N = f_N$ and

$$f * F_N = \frac{1}{N} \sum_{k=0}^{N-1} f_k.$$

Pf By linearity of convolution, it suffices to show that

$$f * e_n = \hat{f}(n) e_n \quad , \text{ and indeed}$$

$$(f * e_n)(x) = \int_0^1 f(t) e_n(x-t)\,dt$$

$$= \int_0^1 f(t) e_{-n}(t)\, e_n(x)\,dt$$

$$= \langle f, e_n \rangle e_n(x)$$

$$= \hat{f}(n) e_n(x) . \quad \square$$

Defn The N-th Cesàro sum of the Fourier series of f is

$$s_N(x) := (f * F_N)(x) .$$

Lemma For $x \in S^1$, $n \geq 0$, and $N \geq 1$, we have

$$D_n(x) = \begin{cases} \sin((2n+1)\pi x)/\sin(\pi x) & \text{if } x \neq 0 \\ 2n+1 & \text{if } x = 0 \end{cases}$$

$$F_N(x) = \begin{cases} \frac{1}{N} \sin^2(N\pi x)/\sin^2(\pi x) & \text{if } x \neq 0 \\ N & \text{if } x = 0. \end{cases}$$

Pf Let $q = e^{\pi i x}$. Then

$$D_n(x) = \sum_{k=-n}^{n} q^{2k} = \begin{cases} (q^{-2n} - q^{2n+2})/(1-q^2) & \text{if } x \neq 0 \\ 2n+1 & \text{if } x = 0. \end{cases}$$

Now $$\frac{q^{-2n} - q^{2n+2}}{1-q^2} \cdot \frac{-q^{-1}}{-q^{-1}} = \frac{q^{2n+1} - q^{-2n-1}}{q - q^{-1}}$$

and $q^k - q^{-k} = 2i\sin(k\pi x)$, giving the result for $D_n(x)$.

Now $$F_N(0) = \frac{1}{N}\sum_{k=0}^{N-1}(2k+1) = \frac{N^2}{N} = N.$$

For $x \neq 0$,

$$F_N(x) = \frac{1}{N}\,\frac{1}{1-q^2}\left(\sum_{n=0}^{N-1} q^{-2n} - \sum_{n=0}^{N-1} q^{2n+2}\right)$$

$$= \frac{1}{N}\cdot\frac{1}{1-q^2}\cdot\left(\frac{1-q^{-2N}}{1-q^{-2}} - \frac{q^2 - q^{2N+2}}{1-q^2}\right)$$

$$= \frac{1}{N} \frac{1}{1-q^2} \left(\frac{1-q^{-2N}}{1-q^{-2}} + \frac{1-q^{2N}}{1-q^{-2}} \right)$$

$$= \frac{1}{N} \cdot \left(\frac{-q^{-2N} + 2 - q^{2N}}{-q^2 + 2 - q^{-2}} \right)$$

$$= \frac{1}{N} \left(\frac{q^N - q^{-N}}{q - q^{-1}} \right)^2$$

$$= \frac{1}{N} \frac{\sin^2(N\pi x)}{\sin^2(\pi x)} . \qquad \square$$

Thm The Fejér kernel F_N is a Dirac kernel.

Pf The expression $F_N(x) = \begin{cases} \frac{1}{N} \sin^2(N\pi x) / \sin^2(\pi x) & \text{if } x \neq 0 \\ N & \text{if } x = 0 \end{cases}$.

demonstrates $F_N \geq 0$. To show $\int_{-1/2}^{1/2} F_N(x)\,dx = 1$, it suffices

to show $\int_{-1/2}^{1/2} D_n(x)\,dx = 1$, which is HW.

To show $\lim_{N\to\infty} \int_{\delta \leq |x| \leq \frac{1}{2}} F_N(x)\,dx = 0$, first prove $F_N(x) \leq \frac{1}{N} \frac{1}{\sin^2(\pi\delta)}$

for $\delta \leq |x| \leq \frac{1}{2}$ (also HW). $\square$

Thm If $\{K_N\}$ is a Dirac kernel and $f \in C^0(S^1)$, then

$$(f * K_N)(x) \xrightarrow[N \to \infty]{} f(x) \text{ uniformly on } S^1.$$

Pf Sketch First prove

Lemma 1 $\forall \varepsilon_1 > 0 \ \exists\, 0 < \delta_1(\varepsilon_1) < \frac{1}{2}$ s.t. $\forall\, 0 < \delta < \delta_1(\varepsilon_1)$, $x \in S^1$, $n \in \mathbb{N}$

$$\int_{-\delta}^{\delta} |f(x-t) - f(x)|\,|K_n(t)|\, dt < \varepsilon_1 .$$

Then prove

Lemma 2 $\forall \delta, \varepsilon_2 > 0 \ \exists N_2(\delta, \varepsilon_2) \in \mathbb{N}$ s.t. $\forall n > N_2(\delta, \varepsilon_2)$, $x \in S^1$

$$\int_{\delta \le |t| \le \frac{1}{2}} |f(x-t) - f(x)|\,|K_n(t)|\, dt < \varepsilon_2 .$$

Now show that $\forall \varepsilon > 0$ $\exists N(f, \varepsilon) \in \mathbb{N}$ s.t. $\forall x \in S^1, n \in \mathbb{Z}$, if $n > N(f, \varepsilon)$, then $|(f * K_n)(x) - f(x)| < \varepsilon$ — which is uniform convergence! $\square$

<u>Cor</u> For $f \in C^0(S^1)$, the Cesàro sum

$$s_N(x) = (f * F_N)(x) = \frac{1}{N} \sum_{n=0}^{N-1} f_n(x)$$

converges uniformly to $f(x)$ on S^1 as $N \longrightarrow \infty$.

<u>Pf</u> F_N is a Dirac kernel, so the thm applies. $\square$

Inversion Thm For $f \in L^2(S^1)$, f_N = N-th Fourier polynomial of f,

$$\lim_{N \to \infty} \|f - f_N\| = 0.$$

Pf Suppose $f \in L^2(S^1)$ and $\varepsilon > 0$. Since $C^0(S^1)$ is dense in $L^2(S^1)$, $\exists\, g \in C^0(S^1)$ with $\|f-g\| < \frac{\varepsilon}{2}$. Since $s_N[g] \longrightarrow g$ uniformly, it also converges to g in L^2 norm. Thus $\exists\, M \in \mathbb{N}$ s.t.

$$\|g - s_M\| < \frac{\varepsilon}{2}.$$

By Δ ineq,

$$\|f - s_M[g]\| \le \|f-g\| + \|g - s_M[g]\| < \varepsilon.$$

By Best Approx'n,

trig poly

$$\|f-f_M\| \le \|f-s_M[g]\| < \varepsilon .$$

Since Fourier polynomials give better approx'ns as $N \to \infty$,

if $N > M$, $\|f-f_N\| \le \|f-f_M\| < \varepsilon$. $\square$

isometry

<u>Cor</u> $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(S^1) \cong \ell^2(\mathbb{Z})$

$$f \longmapsto (\hat{f}(n))_{n\in\mathbb{Z}} . \quad \square$$

<u>Cor</u> For $f, g \in L^2(S^1)$, TFAE: the following are equivalent

1. $f = g$ a.e. on S^1 Lebesgue
2. $\hat{f}(n) = \hat{g}(n)$ $\forall n \in \mathbb{Z}$. $\square$