2025. I. 30

Pf For $v \in V$ let $s_n(v) = \sum_{j=0}^{n} \langle v, e_j \rangle e_j$. Then

$$
\begin{aligned}
0 &\leq \| v - s_n(v) \|^2 \\
&= \left\langle v - \sum_{j=0}^{n} \langle v, e_j \rangle e_j \, , \, v - \sum_{j=0}^{n} \langle v, e_j \rangle e_j \right\rangle \\
&= \|v\|^2 - \sum_{j=0}^{n} |\langle v, e_j \rangle|^2 \qquad \text{(Why?)}
\end{aligned}
$$

so $\sum_{j=0}^{n} |\langle v, e_j \rangle|^2 \leq \|v\|^2 \quad \forall n \in \mathbb{N} \Rightarrow \sum_{j \in \mathbb{N}} |\langle v, e_j \rangle|^2 < \infty$.

Let $m \geq n \in \mathbb{N}$. Then

$$\| s_m(v) - s_n(v) \|^2 = \Big\| \sum_{j=n+1}^{m} \langle v, e_j \rangle e_j \Big\|^2$$

$$= \sum_{j=n+1}^{m} |\langle v, e_j \rangle|^2 \qquad \text{(Pythagoras)}$$

$\Rightarrow$ $(s_n(v))_n$ is Cauchy (why?) so it converges

└ use $\sum_{j\in\mathbb{N}} |\langle v, e_j\rangle|^2 < \infty$

so partial sums of are Cauchy

to $\sum_{j\in\mathbb{N}} \langle v, e_j\rangle e_j \in V$.

For all $j_0 \in \mathbb{N}$, $\langle v, e_{j_0} \rangle = \left\langle \sum_{j\in\mathbb{N}} \langle v, e_j\rangle e_j, e_{j_0} \right\rangle$

$$\Rightarrow \left\langle v - \sum_{j\in\mathbb{N}} \langle v, e_j \rangle e_j , e_{j_0} \right\rangle = 0 \quad \forall j_0 \in \mathbb{N}$$

$$\Rightarrow \quad v = \sum_{j \in \mathbb{N}} \langle v, e_j \rangle e_j \qquad ((e_j) \text{ complete})$$

For the isometry

$$V \longrightarrow \ell^2(\mathbb{N})$$

$$v \longmapsto (\hat{v}(n))_{n \in \mathbb{N}} \quad \text{for } \hat{v}(n) = \langle v, e_n \rangle$$

$$\langle v, v' \rangle = \left\langle \sum_{j \in \mathbb{N}} \langle v, e_j \rangle e_j , \sum_{k \in \mathbb{N}} \langle v', e_k \rangle e_k \right\rangle$$

$$= \sum_{j,k \in \mathbb{N}} \langle v, e_j \rangle \overline{\langle v', e_k \rangle} \langle e_j, e_k \rangle$$

$$= \sum_{j \in \mathbb{N}} \langle v, e_j \rangle \overline{\langle v', e_j \rangle} .$$

$$= \sum_{j \in \mathbb{N}} \hat{v}(j) \overline{\hat{v'}(j)} \qquad \square$$

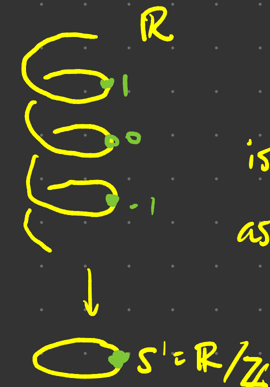
Is $V \to \ell^2(\mathbb{N})$ unitary? Try to produce inverse

$$\ell^2(\mathbb{N}) \longrightarrow V$$

$$c = (c_j)_{j \in \mathbb{N}} \longmapsto \sum_{j \in \mathbb{N}} c_j e_j$$

this works!

Upshot There is only one separable Hilbert space of infinite dimension, $\ell^2(\mathbb{N})$.



$f : S^1 \to \mathbb{C}$ is the same info as a 1-periodic fn on $\mathbb{R}$

L^2

$S^1 := \mathbb{R}/\mathbb{Z}$

Let $X = [a,b]$, S^1, or $\mathbb{R}$.

Set $L^2(X) := \left\{ f : X \to \mathbb{C} \mid f \text{ is Lebesgue measurable}, \int_X |f|^2 < \infty \right\} \Big/ \overset{a.e.}{0}$

Thm $L^2(X)$ with $\langle f, g \rangle = \int_X f \cdot \bar{g}$ is a Hilbert space

and $C^0(X) \subseteq L^2(X)$ is dense.

"cts fns on X"

Goal $(e_n)_{n \in \mathbb{Z}}$ is a complete system for $L^2(S^1)$, $e_n(x) = e^{2\pi i n x}$.

Complex exponential

$z \in \mathbb{C}$

$$e^z := \sum_{n \in \mathbb{N}} \frac{z^n}{n!}$$ has infinite radius of convergence.

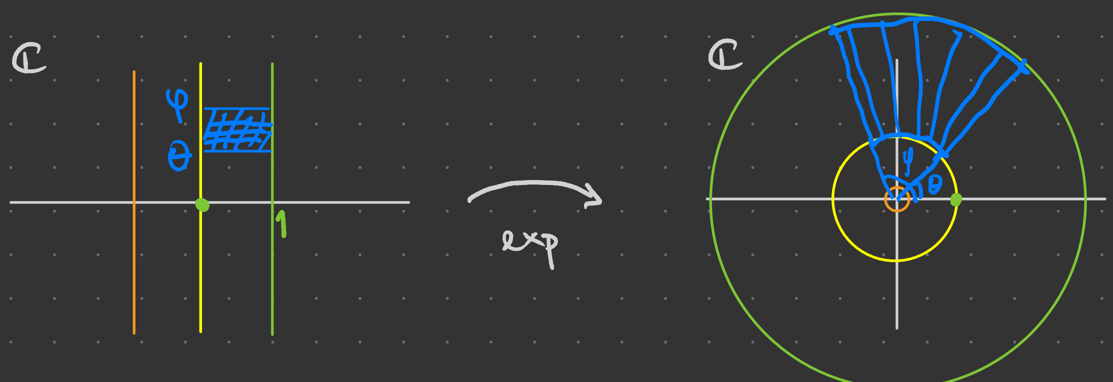
R for $\sum c_n z^n$ is

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

pure imag

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$e^{a+bi} = e^a e^{bi} = e^a (\cos b + i \sin b)$$



Note $\exp : (\mathbb{C}, +) \longrightarrow (\mathbb{C} \setminus \{0\}, \cdot)$ homomorphism

$$e^{z+w} = e^z \cdot e^w$$