2025. I. 29

(Everything you should have learned in Math 201 about)

Inner product spaces (and now will learn really well)

An inner product space $(V, \langle , \rangle)$ is a $\mathbb{C}$-vector space V equipped with a Hermitian form $\langle , \rangle$ that is positive definite.

- linear in first variable
- conjugate symmetric

- $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$ with $\langle x, x \rangle = 0$ iff $x = 0$

$\langle , \rangle : V \times V \to \mathbb{C}$

- $\langle \lambda u + v, w \rangle = \lambda \langle u, w \rangle + \langle v, w \rangle$
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$

E.g. $V = \mathbb{C}^n$, $\langle v, w \rangle = v^T \overline{w} = (v_1\ v_2 \cdots v_n) \begin{pmatrix} \overline{w}_1 \\ \overline{w}_2 \\ \vdots \\ \overline{w}_n \end{pmatrix}$

Defn The norm of $v \in V$ is $\|v\| = \sqrt{\langle v, v \rangle}$.

Cauchy-Schwartz If V is an inner product space, then

$$\forall v, w \in V, \qquad |\langle v, w\rangle| \le \|v\|\,\|w\|.$$

Cor

- $\|\lambda v\| = |\lambda|\,\|v\| \quad \forall \lambda \in \mathbb{C},\ v \in V$ (from sesqui-linearity)
- $\|v+w\| \le \|v\| + \|w\|$. (triangle inequality)

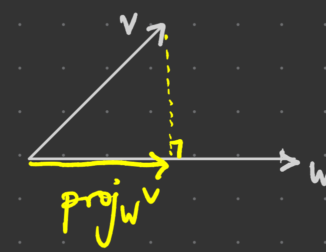
Recall v is orthogonal to w when $\langle v, w\rangle = 0$; in this case write $v \perp w$.

For $S \subseteq V$, the orthogonal complement of S is

$$S^\perp = \{v \in V \mid v \perp s \ \ \forall s \in S\}$$

The orthogonal projection of v onto w is

$$\operatorname{proj}_w v := \underbrace{\frac{\langle v, w\rangle}{\|w\|^2}}_{\text{component of } v \text{ along } w} w$$



Defn A **complete system** in an inner product space V is a family $(a_j)_{j\in J}$ of vectors in V such that $\{a_j\}_{j\in J}^{\perp} = 0$.

formally: $a : J \longrightarrow V$

trivial subspace $\{0\} \leq V$

Call V **separable** when it contains a countable complete system.

- To have $S^\perp = 0$ means that

$$\{v \in V \mid \langle v, s\rangle = 0 \ \forall s \in S\} = 0$$

$$\Leftrightarrow \text{if } \langle v, s\rangle = 0 \ \forall s \in S \text{ then } v = 0.$$

- Note $(v)_{v \in V}$ is always a complete system

separability demands a "small-ish" complete system.

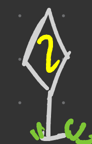
E.g. Bases are complete systems.

Note $\mathbb{C}^n \overset{\text{"}}{=}\text{"} \{f : \{1, 2, \dots, n\} \to \mathbb{C}\}$

E.g. Let $\ell^2(\mathbb{N}) = \{ f : \mathbb{N} \to \mathbb{C} \mid \sum_n |f(n)|^2 < \infty \}$

with $\langle f, g \rangle = \sum_n f(n)\overline{g(n)}$. For $j \in \mathbb{N}$, define $e_j \in \ell^2(\mathbb{N})$

by $e_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{if } n \neq j \end{cases}$



These $e_j \neq x \mapsto e^{2\pi i j x}$

Since $\langle f, e_j \rangle = \sum f(n)\overline{e_j(n)} = f(j)$

we learn that $(e_j)_{j \in \mathbb{N}}$ is a complete system and $\ell^2(\mathbb{N})$ is separable.

Defn An orthonormal system in an inner product space V

is a family $(h_j)_{j\in J}$ of vectors in V such that

$$\langle h_j, h_k \rangle = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j\neq k \end{cases}$$

Kronecker delta

<u>Note</u> $\|h_j\| = \sqrt{\langle h_j, h_j \rangle} = \sqrt{1} = 1$

An orthonormal system that is complete is called an <u>orthonormal basis</u>.

<u>E.g.</u> $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $\ell^2(\mathbb{N})$.

<u>Prop</u> Every separable inner product space admits an orthonormal basis.

Pf Let $(a_j)_{j\in\mathbb{N}}$ be a complete system and apply Gram-Schmidt:

WLOG, every finite subset of a_j is lin ind.

without loss of generality

$$e_0 := \frac{a_0}{\|a_0\|}$$

$$\vdots$$

$$e'_{k+1} := a_{k+1} - \sum_{j=0}^{k} \underbrace{\langle a_{k+1}, e_j\rangle e_j}_{\mathrm{proj}_{e_j} a_{k+1}}$$

$$e_{k+1} := \frac{e'_{k+1}}{\|e'_{k+1}\|}$$

$$\vdots$$

Q Why isn't $e'_{k+1} = 0$?

A By lin ind of a_j

If V is finite dim'l, this terminates to produce a basis.

If V is infinite dim'l, get a sequence $(e_j)_{j\in\mathbb{N}}$ that is orthonormal.

Suppose $\langle h, e_j\rangle = 0 \ \forall j \in \mathbb{N}$. Then $\langle h, a_j\rangle = 0 \ \forall j \in \mathbb{N}$, so $h = 0$, so $(e_j)_j$ is complete as well. $\square$

Thm Suppose V is an infinite-dimensional Hilbert space with orthonormal basis (e_j). Then every element $v \in V$ can be uniquely expressed as

$$v = \sum_{j\in\mathbb{N}} c_j e_j$$

- complete inner prod space: Cauchy sequences converge

A *Cauchy sequence* in V is $(v_i)_{i \in \mathbb{N}}$

such that $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

if $i, j > N$, then $\| v_i - v_j \| < \varepsilon$.

In general, (v_i) *convergent* $\Longrightarrow$ Cauchy.

$\exists v \in V$ s.t.
$\forall \varepsilon > 0 \ \exists N$ s.t.
if $i > N$ then $\| v_i - v \| < \varepsilon$

in this case, say that
$\lim v_i = v$ (in $\| \ \|$).

Completeness is converse.

with the sum convergent in V, and c_j satisfying

$$\sum_{j \in \mathbb{N}} |c_j|^2 < \infty .$$

In fact, $c_j = \langle v, e_j \rangle$ and $v \longmapsto (\langle v, e_j \rangle)_{j \in \mathbb{N}}$ is an isometry $V \longrightarrow \ell^2(\mathbb{N})$.

- linear trans'n $V \xrightarrow{T} W$ that preserves inner products:

$$\langle Tv, Tv' \rangle = \langle v, v' \rangle$$

- necessarily injective

In particular,

$$\langle v, v' \rangle = \sum_{j \in \mathbb{N}} \langle v, e_j \rangle \overline{\langle v', e_j \rangle}$$

and $\|v\|^2 = \sum_{j \in \mathbb{N}} |\langle v, e_j \rangle|^2$.

- if surjective, then admits isometric inverse; called unitary