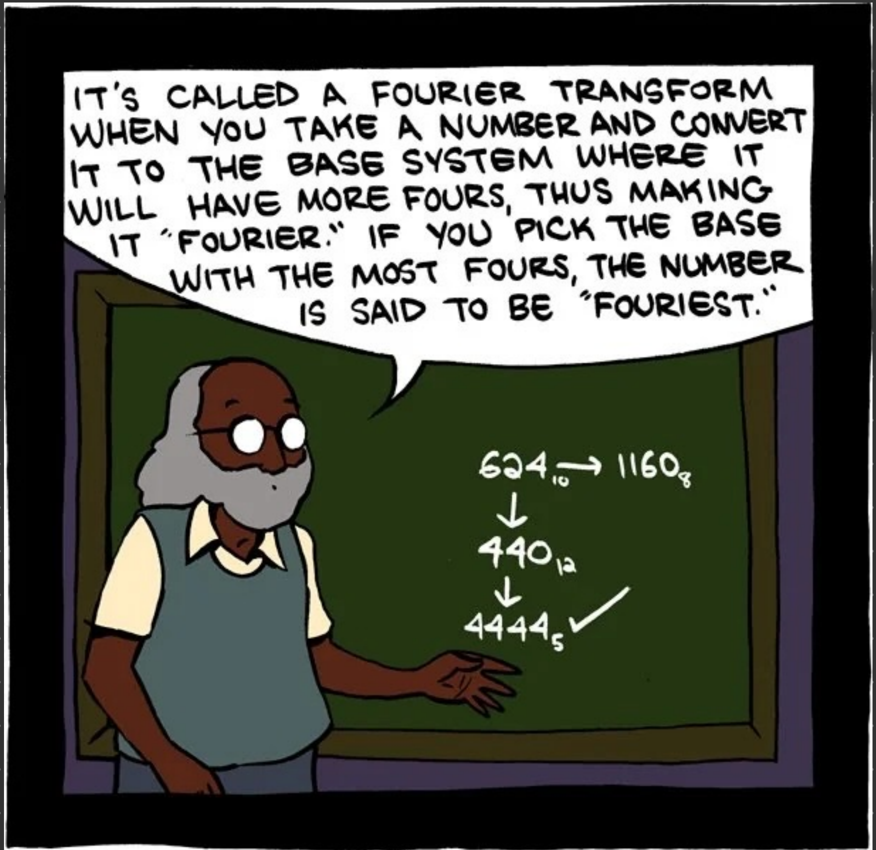
Math 411

Fourier Analysis

Kyle Ormsby
ormsbyk@reed.edu

2025.I.27



Teaching math was way more fun after tenure.

Old idea

- Many $f : \mathbb{R} \to \mathbb{C}$ can be written as a power series

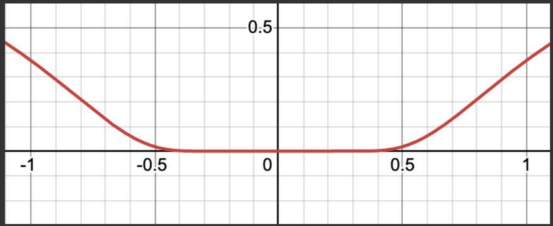
$$f(x) = \sum_{n \geq 0} c_n x^n$$

$c : \mathbb{N} \to \mathbb{C}$ much simpler than f, and

$$c_n = \frac{f^{(n)}(0)}{n!}$$

- But $\sum c_n x^n$ might not converge or it might converge to something other than f

E.g. $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



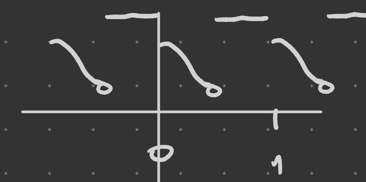
Moral exc $f^{(n)}(0) = 0$ for all n, so $\sum c_n x^n = 0$

- Different functions might have the same Taylor series
- Only C^∞ functions have Taylor series
- Taylor series are locally determined: if $f(x) = g(x)$ for $x \in (-\delta, \delta)$ for some $\delta > 0$, then their Taylor series are the same
- Need f to be smooth (C^∞)

New idea

Suppose $f:\mathbb{R}\to\mathbb{C}$ is 1-periodic : $f(x)=f(x+1)\ \forall x\in\mathbb{R}$.

E.g.



or $f(x)=\cos(2\pi x)$

- Many such functions can be written as

$$f(x)=\sum_{k\in\mathbb{Z}} \overset{=\hat{f}(k)\in\mathbb{C}}{c_k}\, e^{2\pi i k x}$$

Fourier series

where $c_k=\int_0^1 f(x)\,e^{-2\pi i k x}\,dx$.

$e^{(\,)}:\mathbb{C}\to\mathbb{C}$ | $e^{i\theta}=\cos\theta+i\sin\theta$

$e^{z}=\sum_{n\geq 0}\frac{z^n}{n!}$

- $\sum c_k e^{2\pi i k x}$ might not converge, or might converge but not to $f(x)$.
- But: different continuous functions have different Fourier series

- And even some discontinuous functions have convergent Fourier series!
- And Fourier series are determined globally

Demos