24.XI.20

Goals

- Direct sum
- Orthogonal complement
- Orthogonal projection

\oplus

Defn For U, V $\mathbb{F}$-vector spaces, their **direct sum** is

$$U \oplus V := U \times V = \{(u,v) \mid u \in U, v \in V\}$$

with componentwise addition and scalar multiplication:

$$(u,v) + (u',v') = (u+u', v+v'),$$

$$\lambda(u,v) = (\lambda u, \lambda v).$$

E.g. $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$

Prop Let $U, V \leq W$ such that

① $\operatorname{span}(U \cup V) = W$,

② $U \cap V = 0$.

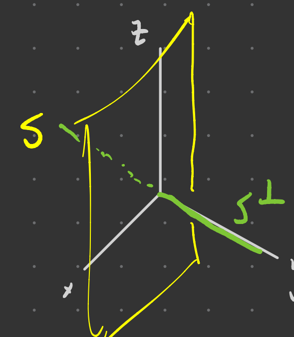
Then
$$U \oplus V \xrightarrow{\cong} W$$
$$(u,v) \longmapsto u+v.$$

Write $W = U \oplus V$

Pf Linear ✓. ① guarantees surjectivity. If $(u,v) \in \ker$ then $u+v=0 \Rightarrow u=-v \in U \cap V$
$\Rightarrow u=v=0$ by ②.
Thus $\ker = 0$ so the map is injective as well. $\square$

From now on, $(V, \langle\, , \rangle)$ an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Defn For $S \subseteq V$, the orthogonal complement of S is
$$S^\perp := \{x \in V \mid \langle x, s\rangle = 0 \ \forall s \in S\}.$$



Q Is $S^\perp$ a subspace of V?

$S = \{s\}$

A $0 \in S^\perp$ ✓. If $x, y \in S^\perp$, $\langle x+\lambda y, s\rangle = \langle x,s\rangle + \lambda\langle y,s\rangle$

$= 0 + \lambda 0 = 0 \quad \forall s \in S \Rightarrow x+\lambda y \in S^\perp$.

Prop Suppose $\dim V = n$, $S = \{v_1, \dots, v_k\} \subseteq V$ is orthonormal.

① S can be extended to an orthonormal basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V.

② If $W = \operatorname{span} S$, then $\{v_{k+1}, \dots, v_n\}$ is an orthonormal basis of $S^\perp = W^\perp$.

③ If $W \leq V$, then $\dim W + \dim W^\perp = \dim V = n$.

④ If $W \leq V$, then $(W^\perp)^\perp = W$. (Similar to $X \setminus (X \setminus Y) = Y$ $\; Y \subseteq X$.)

Pf ① Apply G-S to any basis extension of $\{v_1, \dots, v_k\}$.

② Let $S' = \{v_{k+1}, \dots, v_n\}$, which is lin ind & orthonormal. Since S is orthonormal, $S' \subseteq W^\perp \Rightarrow \operatorname{span} S' \subseteq W^\perp$. For $x \in W^\perp$, since $S \cup S'$ is orthonormal,

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$$

$$= \sum_{i=k+1}^{n} \langle x, v_i \rangle v_i \qquad [x \in W^\perp]$$

$$\in \operatorname{span} S'$$

So $\operatorname{span} S' = W^\perp \Rightarrow S'$ is a basis for $W^\perp$.

(3) Follows directly from (2).

(4) We have $(W^\perp)^\perp = \{x \in V \mid \langle x, y \rangle = 0 \text{ for } y \in W^\perp\} \supseteq W$.

Now $\dim (W^\perp)^\perp \underset{(3)}{=} n - \dim W^\perp \underset{(3)}{=} \dim W$ so $W = (W^\perp)^\perp$. $\square$

\perp

Prop Let $W \leq V$. Then $V = W \oplus W^\perp$, i.e. for all $y \in V$ there is a unique $u \in W, v \in W^\perp$ such that $y = u + v$.

Call u the orthogonal projection of y onto W. If $\{u_1, \dots, u_k\}$ is an orthonormal basis for W, then $u = \sum_{i=1}^{k} \langle y, u_i \rangle u_i$.

PF Let $\{u_1, \dots, u_k\}$ be an orthonormal basis for W and define

$u = \sum_{i=1}^{k} \langle y, u_i \rangle u_i$, $v = y - u$. Then $u \in W$ and $y = u + v$.

Furthermore, for $1 \leq j \leq k$,

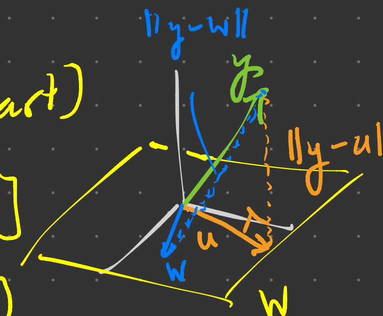
$$\begin{aligned}
\langle v, u_j \rangle &= \langle y - u, u_j \rangle \\
&= \langle y, u_j \rangle - \left\langle \sum_{i=1}^{k} \langle y, u_i \rangle u_i, u_j \right\rangle \\
&= \langle y, u_j \rangle - \sum_{i=1}^{k} \langle y, u_i \rangle \langle u_i, u_j \rangle \\
&= \langle y, u_j \rangle - \langle y, u_j \rangle \cdot 1 \qquad \text{[orthonormality]} \\
&= 0
\end{aligned}$$

so $v \in W^\perp$. Moral exercise: check $W \cap W^\perp = 0$ and thus the expression is unique. $\square$

Cor The orthogonal projection u of y onto W is the vector in W closest to y: $\|y-u\| \le \|y-w\|$ for all $w \in W$ with equality iff $u=w$.

Pf Write $y = u+v$ with $u \in W$, $v \in W^\perp$. Take $w \in W$. Then $u - w \in W$, $y - u \in W^\perp$ so by Pythagoras,

$$\|y-w\|^2 = \|y-u + u-w\|^2 \quad \text{[Get Smart]}$$
$$= \|y-u\|^2 + \|u-w\|^2 \quad \text{[Pythag]}$$
$$\geq \|y-u\|^2 \quad [\|u-w\|^2 \geq 0]$$



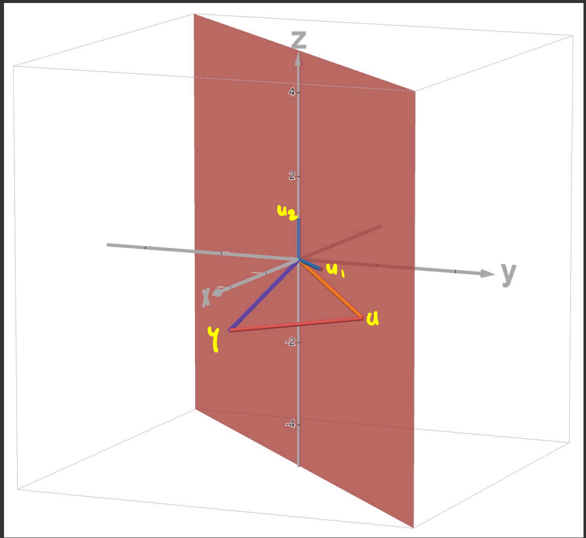
with equality iff $\|u-w\| = 0$ iff $u = w$. $\square$

E.g. Let $W = \text{span}\{\underset{u_1}{(1,1,0)}, \underset{u_2}{(0,0,1)}\}$. Determine the distance of $y = (4,0,-1)$ from W:

$$u = \frac{\langle y, u_1 \rangle}{\|u_1\|^2} u_1 + \frac{\langle y, u_2 \rangle}{\|u_2\|^2} u_2 = 2\sqrt{2}\, u_1 + \frac{-\sqrt{2}}{2} u_2$$

$$= \left(2\sqrt{2},\, 2\sqrt{2},\, -\frac{\sqrt{2}}{2}\right)$$

so $y - u = \left(4 - 2\sqrt{2},\, -2\sqrt{2},\, -1 + \frac{\sqrt{2}}{2}\right)$ with $\|y - u\| = \sqrt{\frac{67}{2} - 17\sqrt{2}} \approx 3.075$



Topics

- Spectral theorem
- Markov chains $\longrightarrow$ PageRank
- cross product
- matrix groups $SO(n)$ (esp topology $SO(3)$)
  $SL_2\mathbb{R}$
- normed division algebras : quaternions
- SVD - singular value decomposition
- your ideas!!