24. XI. 18

Goals

- Orthonormality
- Gram-Schmidt algorithm

$(V, \langle\, , \rangle)$ an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

$u \perp v$

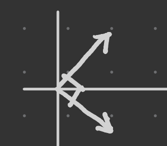
Defn For $S \subseteq V$, S is orthogonal if $\langle u, v \rangle = 0$ $\forall u \neq v \in S$;

it is orthonormal when, additionally, $\langle u, u \rangle = 1$ $\forall u \in S$.

$\|u\| = 1$

E.g. ① $\{e_1, \dots, e_n\}$ is orthonormal in F^n.

② $\left\{ \frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1) \right\}$ is orthonormal in F^2.

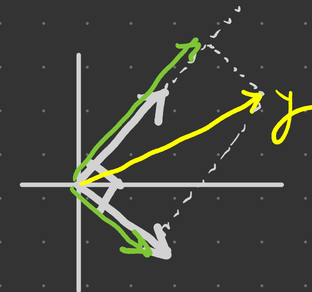


③ $\{2\sin(2\pi x), 2\cos(2\pi x)\}$ is orthonormal in $C_{\mathbb{R}}[0,1]$

$$\int_0^1 (2\sin(2\pi x))^2 \, dx = \int_0^1 (2\cos(2\pi x))^2 \, dx = 1, \quad \int_0^1 2\sin(2\pi x) \cdot 2\cos(2\pi x) \, dx$$

Prop Let $S=\{v_1,\dots,v_k\}\subseteq V$ be orthogonal. Then for $y\in \operatorname{Span} S$,

$$y=\sum_{j=1}^{k} \underbrace{\frac{\langle y,v_j\rangle}{\|v_j\|^2}\, v_j}_{\text{proj'ns of } y \text{ onto } v_j}.$$



Pf Say $y=\sum_{i=1}^{k}\lambda_i v_i$. Then

$$\begin{aligned}
\langle y, v_j\rangle &= \Big\langle \sum_i \lambda_i v_i ,\, v_j\Big\rangle \\
&= \sum_i \lambda_i \langle v_i, v_j\rangle && [\langle\,,\rangle \text{ linear in 1st var}] \\
&= \lambda_j \langle v_j, v_j\rangle && [\langle v_i, v_j\rangle = 0 \quad i\neq j] \\
&= \lambda_j \|v_j\|^2 .
\end{aligned}$$

Thus $\lambda_j = \frac{\langle y, v_j \rangle}{\|v_j\|^2} \Rightarrow y = \sum_j \frac{\langle y, v_j \rangle}{\|v_j\|^2} v_j$. $\square$

<u>Cor</u> If $S \subseteq V$ is orthonormal and $y \in \operatorname{span} S$, then

$$y = \sum_{j=1}^{k} \langle y, v_j \rangle v_j . \quad \square \qquad (\|v_j\|^2 = 1)$$

<u>Cor</u> If $S \subseteq V$ is an orthogonal set of vectors, then S is linearly independent.

WTS: $\lambda_i = 0$

<u>Pf</u> Let $S = \{v_1, \dots, v_k\}$ and suppose $\sum \lambda_i v_i = 0$. Then

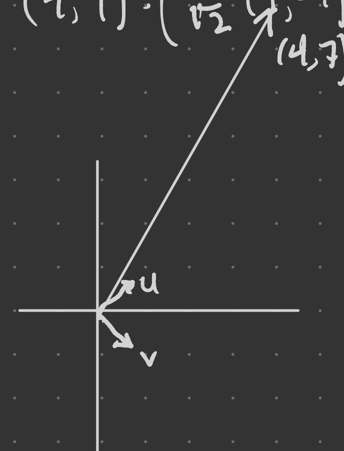
$$0 = \langle 0, v_j \rangle = \left\langle \sum_i \lambda_i v_i , v_j \right\rangle = \sum_i \lambda_i \langle v_i, v_j \rangle = \lambda_j \langle v_j, v_j \rangle .$$

Since $\langle v_j, v_j \rangle \neq 0$, we have $\lambda_j = 0$. $\square$

E.g. Let $\beta = \left(\overset{u}{\overset{\shortparallel}{\frac{1}{\sqrt{2}}(1,1)}}, \overset{v}{\overset{\shortparallel}{\frac{1}{\sqrt{2}}(1,-1)}} \right)$, an orthonormal basis for $\mathbb{R}^2$.

Let $y = (4,7)$. What are the β-coordinates of y?

We have
$$
\begin{aligned}
y &= \langle y, u \rangle u + \langle y, v \rangle v \\
&= (4,7) \cdot \left(\frac{1}{\sqrt{2}}(1,1) \right) u + (4,7) \cdot \left(\frac{1}{\sqrt{2}}(1,-1) \right) v \\
&= \frac{11}{\sqrt{2}} u - \frac{3}{\sqrt{2}} v .
\end{aligned}
$$



Goal Transform a linearly independent set into an orthonormal set with the same span.

$$\boxed{\{(1,2),(0,-1)\}}$$

Algorithm [Gram-Schmidt]

Input: Lin ind set $S = \{w_1, \dots, w_n\} \subseteq V$.

(1) Let $v_1 = w_1$.

$(2, \dots, n)$ For $k = 2, \dots, n$ define

$$v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} v_i .$$

Think of as "straightening" w_k wrt $v_1, \dots, v_{k-1}$ by removing proj'ns onto them

Output: $S' = \{v_1, \dots, v_n\} \subseteq V$ orthogonal with span S' = span S.

or

Output: $S'' = \left\{ \frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|} \right\} \subseteq V$ orthonormal with span S'' = span S.

Validity Pf [Idea] Use induction to check that the "straightening" operation preserves span and induces orthogonality.

(Full details below.) $\square$

Cor Every finite dimensional inner product space has an orthonormal basis. $\square$

E.g. Take $V = \mathbb{R}[x]_{\leq 1}$ with inner product $\langle f, g \rangle = \int_0^1 f \cdot g$.

Let's apply Gram-Schmidt to $\{1, x\}$:

(1) $v_1 = 1$

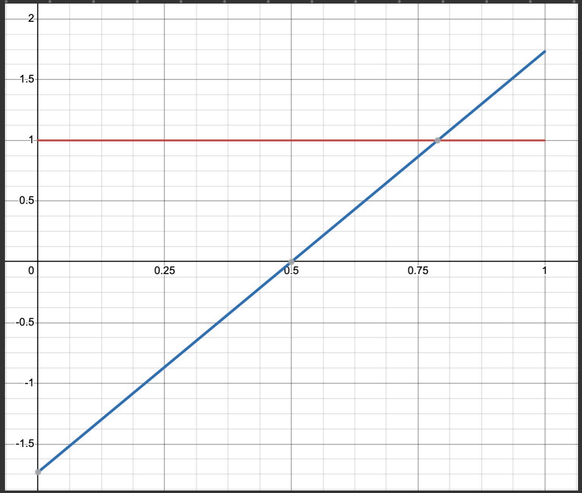
(2) $v_2 = x - \dfrac{\langle x, v_1 \rangle}{\|v_1\|^2} v_1 = x - \dfrac{\langle x, 1 \rangle}{\|1\|^2} 1$

$$= x - \int_0^1 x\,dx \cdot 1 = x - \tfrac{1}{2}.$$

Normalizing: $\|v_1\| = \sqrt{\int_0^1 1^2\,dx} = 1.$

$$\|v_2\| = \sqrt{\int_0^1 (x - 1/2)^2\,dx} = \sqrt{1/12}$$

So $\{1, \sqrt{12}\,(x - 1/2)\}$ is an orthonormal basis of $\mathbb{R}[x]_{\leq 1}$.



Q How can we interpret orthonormality of these visually?

Problem Apply G-S to $\{(1,2), (0,-1)\}$ in $\mathbb{R}^2$.

A

$v_1 = (1,2)$

$$v_2 = (0,-1) - \frac{(0,-1)\cdot(1,2)}{\|(1,2)\|^2}(1,2)$$

$$= (0,-1) - \frac{-2}{5}(1,2)$$

$$= (0,-1) + \left(\frac{2}{5}, \frac{4}{5}\right)$$

$$= \left(\frac{2}{5}, \frac{-1}{5}\right).$$

Check: $(1,2)\cdot\left(\frac{2}{5}, \frac{-1}{5}\right) = \frac{2}{5} + 2\left(\frac{-1}{5}\right) = 0$

So $S' = \left\{(1,2), \left(\frac{2}{5}, \frac{-1}{5}\right)\right\}$

$$\left\|\left(\frac{2}{5}, \frac{-1}{5}\right)\right\| = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5}$$

$$S'' = \left\{\frac{1}{\sqrt{5}}(1,2), \frac{\sqrt{5}}{5}(2,-1)\right\}$$

Proof of validity of the algorithm. We prove this by induction on n. The case $n = 1$ is clear. Suppose the algorithm works for some $n \geq 1$. and let $S = \{w_1, \ldots, w_{n+1}\}$ be a linearly independent set. By induction, running the algorithm on the first n vectors in S produces orthogonal $v_1, \ldots, v_n$ with

$$\operatorname{Span}\{v_1, \ldots, v_n\} = \operatorname{Span}\{w_1, \ldots, w_n\}.$$

Running the algorithm further produces

$$v_{n+1} = w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i.$$

It cannot be that $v_{n+1} = 0$, since otherwise, the above equation we would say

$$w_{n+1} \in \operatorname{Span}\{v_1, \ldots, v_n\} = \operatorname{Span}\{w_1, \ldots, w_n\},$$

contradicting the assumption of the linear independence of the w_i. So $v_{n+1} \neq 0$.

We now check that v_{n+1} is orthogonal to the previous v_i. For $j = 1, \ldots, n$, we have

$$\begin{aligned}
\langle v_{n+1}, v_j \rangle &= \left\langle w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i, v_j \right\rangle \\
&= \langle w_{n+1}, v_j \rangle - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle \\
&= \langle w_{n+1}, v_j \rangle - \frac{\langle w_{n+1}, v_j \rangle}{\|v_j\|^2} \langle v_j, v_j \rangle \\
&= \langle w_{n+1}, v_j \rangle - \langle w_{n+1}, v_j \rangle \\
&= 0.
\end{aligned}$$

We have shown $\{v_1, \ldots, v_{n+1}\}$ is an orthogonal set of vectors, and we would now like to show that its span is the span of $\{w_1, \ldots, w_{n+1}\}$. First, since $\{v_1, \ldots, v_{n+1}\}$ is orthogonal, it's linearly independent. Next, we have

$$\operatorname{Span}\{v_1, \ldots, v_{n+1}\} \subseteq \operatorname{Span}\{v_1, \ldots, v_n, w_{n+1}\} \subseteq \operatorname{Span}\{w_1, \ldots, w_n, w_{n+1}\}.$$

Since $\operatorname{Span}\{v_1, \ldots, v_{n+1}\}$ is an $(n+1)$-dimensional subspace of the $(n+1)$-dimensional space $\operatorname{Span}\{w_1, \ldots, w_n, w_{n+1}\}$, these spaces must be equal. $\square$