24. XI. 11

Goals
- Powers of diagonalizable matrices
- Graphs via matrices
- Matrix powers and counting walks on graphs.

Suppose $A \in F^{n \times n}$ is diagonalizable. Then $A = P^{-1} D P$ for $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $P \in GL_n(F) := \{ Q \in F^{n \times n} \mid Q \text{ invertible} \}$

Powers of A?

$$\begin{aligned} A^2 &= (P^{-1} D P)(P^{-1} D P) \\ &= P^{-1} D (P P^{-1}) D P \\ &= P^{-1} D^2 P = P^{-1} \mathrm{diag}(\lambda_1^2, \ldots, \lambda_n^2) P \end{aligned}$$

In general,

$$A^k = P^{-1} D^k P$$

$$= P^{-1} \operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k) P \quad \text{by induction.}$$

Q What about non-diag'le matrices? $P^{-1} J^k P$, J in Jordan form

Graphs via matrices

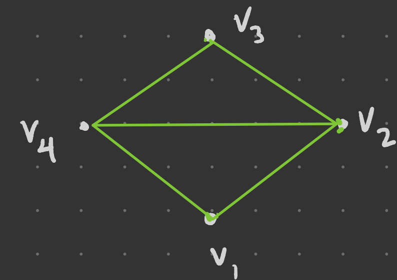
A simple graph is $G = (V, E)$, V = vertices

$E \subseteq \binom{V}{2} = \{\{v, w\} \mid v \neq w \in V\}$

E.g. $V = \{v_1, v_2, v_3, v_4\}$

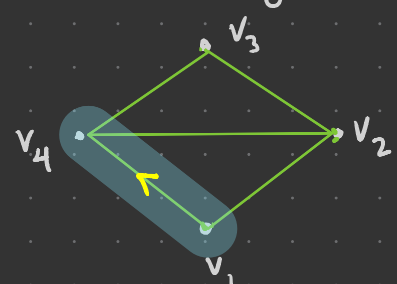
$E = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\}$

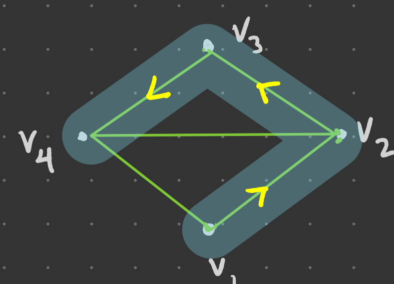
Visually:



A walk (of length ℓ) in G is a sequence of vertices $u_0 u_1 \cdots u_\ell$ such that $\{u_i, u_{i+1}\} \in E$, $0 \le i \le \ell - 1$.

E.g. In above graph, $v_1 v_4$ and $v_1 v_2 v_3 v_4$ are walks from v_1 to v_4 of lengths 1 and 3, respectively.



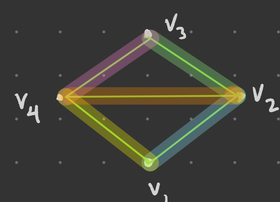


Defn Let $G=(V,E)$ be a graph with $V=\{v_1,\dots,v_n\}$.

The adjacency matrix of G is the $n\times n$ matrix

$A = A(G) \in \mathbb{R}^{n\times n}$ with

$$A_{ij} = \begin{cases} 1 & \{v_i,v_j\}\in E \\ 0 & \{v_i,v_j\}\notin E. \end{cases}$$

E.g. $A\Big($ 

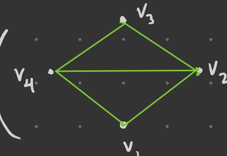
$\Big) =$

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & & & & \\ v_2 & & & & \\ v_3 & & & & \\ v_4 & & & & \end{array}
\quad
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Note Depends on choice of ordering of V.

Thm If $A = A(G)$, then the number of walks from v_i to v_j of length ℓ in G is $(A^\ell)_{ij}$.

Pf HW! $\square$

Problem For $A = A\Big($ 

$\Big)$ compute A^2, A^3 to determine (a) # length 2 walks v_2 to v_3,

(b) # length 3 walks v_2 to v_2.

Then find all such walks.

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & & & & \\ v_2 & & & & \\ v_3 & & & & \\ v_4 & & & & \end{array}$$

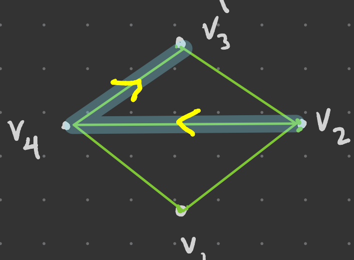
$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Defn A matrix A is symmetric when $A = A^T$.

Note If $A = A(G)$, then A is symmetric.

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$$



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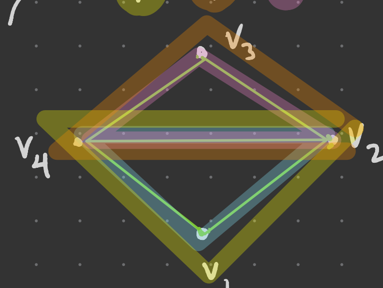


Thm If A is an $n \times n$ symmetric matrix with real entries, then A is diagonalizable.

Pf This is a corollary of the "Spectral Theorem". $\square$

Upshot Adjacency matrices are diagonalizable!

Cor Given a graph G with n vertices and $0 \le i, j \le n$ $\exists\, c_1, \dots, c_n \in \mathbb{R}$ (independent of ℓ) such that

depend on i, j

$$\#\text{walks } v_i \text{ to } v_j \text{ of length } \ell \text{ in } G = \sum_{r=1}^{n} c_r \lambda_r^{\ell} \qquad \square$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of $A(G)$ (counted w/ multiplicity)

Idea # length l walks v_i to v_j $= (A^l)_{ij}$

$$= \left(P^{-1}\,\mathrm{diag}(\lambda_1^l, \ldots, \lambda_n^l)\,P\right)_{ij}$$

Defn A walk is **closed** when it starts and ends at the same vertex.

Cor The number of length ℓ closed walks in G is
$$\operatorname{tr}\left(A(G)^{\ell}\right).$$
$\square$

Prop For $A \in F^{n \times n}$, $\operatorname{tr}(A) =$ sum of eigenvalues of A counted according to algebraic multiplicity.

Note If $\chi_A(x) = c(x-\lambda_1)\cdots(x-\lambda_n)$, then $\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

This sum is in F (b/c $\operatorname{tr}(A) \in F$) even if λ_i are not!

Pf of Prop Let $\bar{F}$ = algebraic closure of F.

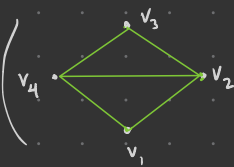
$\exists P \in GL_n(\bar{F})$ such that $P^{-1}AP = J$ is in Jordan form.

The diagonal of J is $\lambda_1, \ldots, \lambda_n$. Now

$$\begin{aligned} tr(A) &= tr(PJP^{-1}) \\ &= tr(PP^{-1}J) \qquad [\, tr(UV) = tr(VU) \,] \\ &= tr(J) \\ &= \sum_{i=1}^{n} \lambda_i . \end{aligned}$$

$\square$

Cor Suppose $A(G) \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ listed with algebraic multiplicity. Then the number of closed walks in G is $\lambda_1^\ell + \cdots + \lambda_n^\ell$.

E.g. If $A = A\Bigg($ 

$\Bigg)$, then

$$\chi_A(x) = x(x+1)(x^2 - x - 4)$$

with roots $0, -1, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$.

Thus the # closed walks in G of length ℓ is

$$w(\ell) = 0^\ell + (-1)^\ell + \left(\frac{1+\sqrt{17}}{2}\right)^\ell + \left(\frac{1-\sqrt{17}}{2}\right)^\ell$$

where $0^\ell = \begin{cases} 1 & \ell = 0 \\ 0 & \ell > 0 \end{cases}$.

ℓ	0	1	2	3	4	5	6
$W(\ell)$	4	0	10	12	50	100	298

spectral graph theory!