24.XI.15

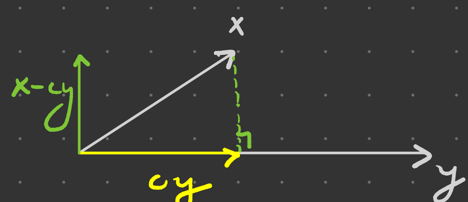
Goals

- Components
- (Orthogonal) projection
- Cauchy-Schwarz & triangle inequalities
- Angles

Fix $(V, \langle , \rangle)$ an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Recall $x, y \in V$ are orthogonal (written $x \perp y$) when $\langle x, y \rangle = 0$.

Provocation Given $x, y \in V$ is there $c \in F$ such that $x - cy \perp y$?



Answer $\langle x - cy, y \rangle = 0 \iff \langle x, y \rangle - c\langle y, y \rangle = 0$

$$\iff c\langle y, y \rangle = \langle x, y \rangle$$

$$\iff c = \frac{\langle x, y \rangle}{\langle y, y \rangle} = \frac{\langle x, y \rangle}{\|y\|^2}$$

for $y \neq 0$

Defn The component of x along $y^{\neq 0}$ is

$$c = \frac{\langle x, y \rangle}{\|y\|^2}$$

and the (orthogonal) projection of x to y is the vector

$$cy = \frac{\langle x, y \rangle}{\|y\|^2} y.$$

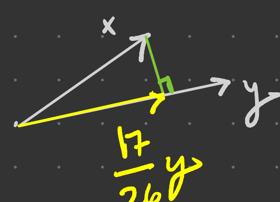
E.g. (1) If $V = F^n$ with ordinary inner product, then for $x \in V$

$$\frac{\langle x, e_j \rangle}{\|e_j\|^2} = x_j \quad \text{for } x = (x_1, \dots, x_n).$$

The projection of x to e_j is $x_j e_j$.

(2) $x = (3,2)$, $y = (5,1)$ in $\mathbb{R}^2$ with ordinary inner product.

Then $$\frac{\langle x, y \rangle}{\|y\|^2} = \frac{3\cdot 5 + 2\cdot 1}{5\cdot 5 + 1\cdot 1} = \frac{17}{26} \approx 0.65$$



(3) $f(x)=\sin(x),\ g(x)=x \in C_{\mathbb{R}}[0,\pi/2]$ with $\langle f,g\rangle = \int_0^{\pi/2} f\cdot g$

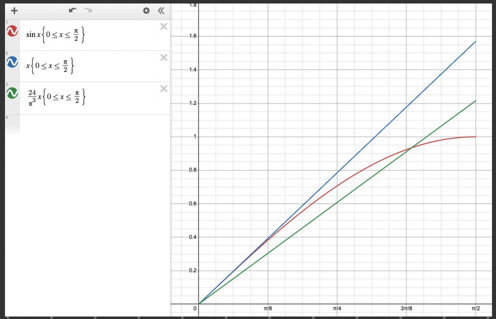
$$\langle f,g\rangle = \int_0^{\pi/2}\sin(x)\cdot x\,dx = -x\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx$$

$u=x \quad dv=\sin x\,dx$

$du=dx \quad v=-\cos x\,dx$

$$=0+\sin x\Big|_0^{\pi/2}$$

$$=1$$



$$\|g\|^2 = \int_0^{\pi/2} x^2\,dx = \frac{x^3}{3}\Big|_0^{\pi/2} = \frac{\pi^3}{24}$$

$$\frac{\langle f,g\rangle}{\|g\|^2} = \frac{24}{\pi^3}\approx 0.774$$ and the proj'n of $\sin x$ onto x is

$$\frac{24}{\pi^3}x.$$

Thm For $x, y \in V$, $\lambda \in F$,

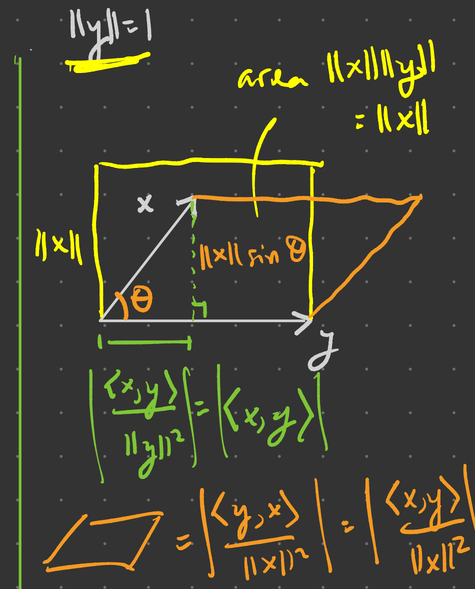
??

① $\|\lambda x\| = |\lambda| \, \|x\|$

② $\|x\| = 0 \iff x = 0$

③ [Cauchy-Schwarz] $\boxed{|\langle x, y\rangle| \le \|x\|\|y\|}$

④ [triangle inequality] $\|x+y\| \le \|x\| + \|y\|$



Pf ①, ② ✓

③ If $y = 0$, we're done as $0 \le 0$. For $y \ne 0$, let

$$c = \frac{\langle x, y\rangle}{\|y\|^2}.$$

Then $x - cy \perp y \Rightarrow x - cy \perp cy$.

By Pythagoras, $\|x-cy\|^2 + \|cy\|^2 = \|x - cy + cy\|^2 = \|x\|^2$.

$$\Rightarrow \|cy\|^2 \leq \|x\|^2 \qquad [\text{since } \|x-cy\|^2 \geq 0].$$

Taking $\sqrt{\ }$: $$\|x\| \geq \|cy\| = |c|\,\|y\| = \left|\frac{\langle x,y\rangle}{\|y\|^2}\right| \|y\|$$

increasing

$$= \left|\frac{\langle x,y\rangle}{\|y\|}\right|$$

$$\Rightarrow \|x\|\,\|y\| \geq |\langle x,y\rangle| \text{ as desired.}$$

④ Facts (i) $z+\bar{z} = 2\,\mathrm{Re}(z)$ (ii) $\mathrm{Re}(z) \leq |z|$

Now observe $$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle$$
$$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} \qquad [\text{conj symm}]$$

$$= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x,y\rangle) \qquad [(i)]$$

$$\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| \qquad [(ii)]$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \qquad [\text{Cauchy-Schwarz}]$$

$$= (\|x\| + \|y\|)^2 .$$

Taking $\sqrt{\ }$ $\Rightarrow$ triangle inequality. $\square$

Defn The distance between $x, y \in V$ is $d(x,y) := \|x-y\|$.

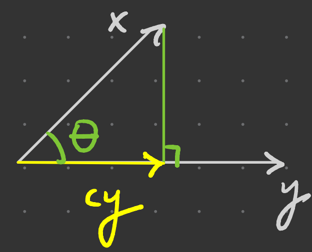
This makes (V, d) a metric space:

- $d(x,y) = d(y,x)$
- $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$

- $d(x,z) \le d(x,y) + d(y,z)$.

<u>Note</u> There are (many!) metrics not induced by inner products/norms.

<u>Angles</u> Idea:



$$\cos\theta = \frac{\|cy\|}{\|x\|} = |c|\frac{\|y\|}{\|x\|}$$

$$= \frac{|\langle x,y\rangle|}{\|y\|^2}\frac{\|y\|}{\|x\|}$$

$$= \frac{|\langle x,y\rangle|}{\|x\|\,\|y\|}.$$

Dropping $|\ |$ in numerator makes this work in all quadrants, so ...

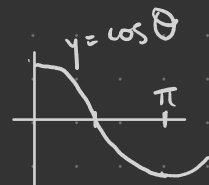
Defn Let $(V, \langle , \rangle)$ be an inner product space over $\mathbb{R}$.

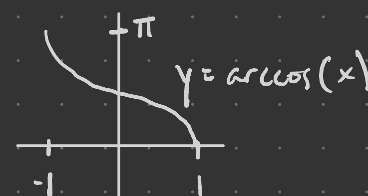
Then **angle** θ between $x, y \in V$ is

$$\theta := \arccos\left(\frac{\langle x, y \rangle}{\|x\| \, \|y\|} \right).$$

Note

- $\cos\theta = \dfrac{\langle x, y\rangle}{\|x\|\|y\|}$ and $\langle x, y \rangle = \|x\| \, \|y\| \cos\theta$

- 



By Cauchy-Schwarz, $\dfrac{|\langle x, y\rangle|}{\|x\| \, \|y\|} \leq 1$

$$\Rightarrow \quad -1 \le \frac{\langle x, y\rangle}{\|x\|\,\|y\|} \le 1 \quad \text{so arccos makes sense.}$$

- Also have $\theta = \arccos\left(\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle\right)$

$\frac{v}{\|v\|}$ is the unit vector in the direction of v.