Day 28

Learning Goals

- Exponential matrices
- Solving linear systems of differential eq'ns

Suppose we're making bread dough and

$$x(t) = \text{amount of yeast at time } t$$

and the growth rate of the yeast is proportional to the amount of yeast:

$$x'(t) = \lambda x(t)$$

for some constant $\lambda \in \mathbb{R}$.

Then $\dfrac{x'(t)}{x(t)} = \lambda$ and, integrating,

$$\int \frac{x'(t)}{x(t)} dt = \int \lambda \, dt \implies \log(x(t)) = \lambda t + b$$

natural logarithm

for some constant b. Exponentiating:

$$x(t) = e^{\lambda t + b} = c e^{\lambda t}$$

for some constant $c = e^b$. Evaluating at $t = 0$,

$$x(0) = c e^0 = c$$

so c = initial amt of yeast.

Now consider a 2-dim'l system

$x_1(t)$ = # frogs in a pond at time t

$x_2(t)$ = # flies in a pond at time t

and suppose

$$\left.\begin{aligned} x_1'(t) &= a\,x_1(t) + b\,x_2(t) \\ x_2'(t) &= c\,x_1(t) + d\,x_2(t) \end{aligned}\right\}$$

Does this make sense biologically? (Sure...)

for some constants $a, b, c, d \in \mathbb{R}$.

Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$. Then

$$x'(t) = A\,x(t)$$

$$\cancel{x(t) = A\,x'(t)} \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thm Let $A \in \mathrm{Mat}_{n \times n}(F)$ for $F = \mathbb{R}$ or $\mathbb{C}$ and suppose $x(t) = A\,x'(t)$ and $x(0) = p$. Then

$$x(t) = \underbrace{e^{At}}_{\substack{\text{exponential matrix:} \\ \text{will define presently}}}\ \underbrace{p}_{\substack{\text{initial condition} \\ \text{(as column vector)}}}.$$

Recall that for $a \in \mathbb{C}$,

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k ,$$

a series that converges everywhere.

Define

$$e^{At} := \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k = I_n + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \frac{1}{24}A^4t^4 + \cdots .$$

So each entry of e^{At} is a power series in t that happens to converge for all t!

We can compute e^{At} via diagonalization:

For A diagonalizable,

$$P^{-1}AP = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) .$$

Then $A^k = (PDP^{-1})^k = PD^kP^{-1} = P \operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k) P^{-1}$.

Thus (modulo some convergence details)

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(PD^kP^{-1})t^k$$

$$= Pe^{Dt}P^{-1} .$$

Since D is diagonal, a short comp'n shows

$$e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$$

so

$$\boxed{e^{At} = P\,\mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})P^{-1}.}$$

E.g.

$$x_1' = x_2 \qquad \text{so } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$x_2' = x_1$$

with $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Therefore

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

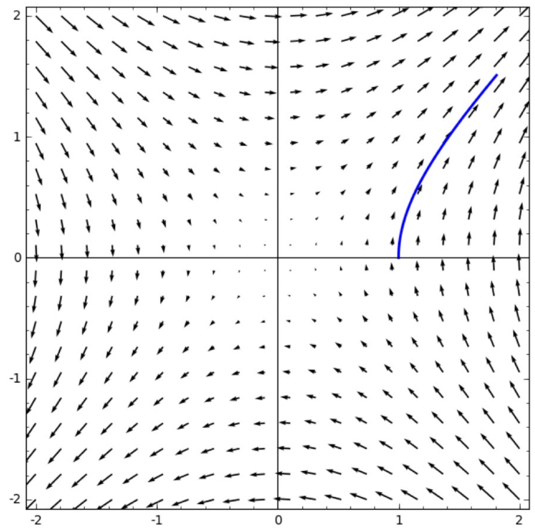
$$= \frac{1}{2}\begin{pmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

If $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then

$$x(t) = e^{At}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.$$

Here's a plot with arrows representing x' at a particular x (velocity field) and the blue curve giving $t \mapsto x(t)$, $t \geq 0$:



(Note the behavior of x' along the "eigenaxes".)

E.g. Now consider the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned} \quad \text{so} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with}$$

$\chi_A(x) = x^2 + 1$ — only diagonalizable over $\mathbb{C}$!

Still, $P^{-1}AP = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ with $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

and $e^{At} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ \frac{1}{2}i & \frac{1}{2} \end{pmatrix}$

$e^{it} = \cos t + i \sin t$

$$= \frac{1}{2} \begin{pmatrix} e^{it} + e^{-it} & ie^{-it} - ie^{it} \\ ie^{it} - ie^{-it} & e^{it} + e^{-it} \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

If $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then

$$x(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos t + \sin t \\ -\sin t + \cos t \end{pmatrix}.$$

Visually,