24.XI.8

Goals
- Eigenvectors with distinct eigenvalues are linearly independent
- Algebraic & geometric multiplicity
- Jordan normal form

Recall If $f: V \longrightarrow V$ linear, then $\chi_f(x) = \det(A_\alpha^\alpha(f) - x \cdot I_n)$ for α some/any ordered basis of V and $n = \dim V$.

Prop Suppose $f: V \longrightarrow V$ linear with eigenvectors $v_1, \ldots, v_k \in V$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_k$, $\lambda_i \neq \lambda_j$ for $i \neq j$. Then $v_1, \ldots, v_k$ are linearly independent.

Pf Proceed by induction on k.

k=1 Eigenvectors are nonzero. ✓

k>1 For induction, suppose $v_1, \dots, v_{k-1}$ are lin ind and that

$$\mu_1 v_1 + \dots + \mu_k v_k = 0. \quad (\star)$$

WTS $\mu_1 = \dots = \mu_k = 0$

Apply $f - \lambda_k \, id_V$ to this expression:

$$(f - \lambda_k id_V)(\mu_1 v_1 + \dots + \mu_k v_k) = (f - \lambda_k id_V)(0)$$

$$\Rightarrow \mu_1 f(v_1) + \dots + \mu_k f(v_k) - (\mu_1 \lambda_k v_1 + \dots + \mu_k \lambda_k v_k) = 0$$

$$\Rightarrow \mu_1 \lambda_1 v_1 + \dots + \mu_k \lambda_k v_k - (\mu_1 \lambda_k v_1 + \dots + \mu_k \lambda_k v_k) = 0$$

[v_i eigen for f]

$$\Rightarrow \underbrace{\left(\mu_1 (\lambda_1 - \lambda_k) v_1 + \dots + \mu_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1}\right)}_{=0} + \mu_k \underbrace{\cancel{(\lambda_k - \lambda_k)}}_{0} v_k = 0$$

$$\Rightarrow \mu_1 (\lambda_1 - \lambda_k) = \dots = \mu_{k-1} (\lambda_{k-1} - \lambda_k) = 0$$

[lin ind $v_1, \dots, v_{k-1}$]

$\Rightarrow \mu_1 = \cdots = \mu_{k-1} = 0$ [λ_i distinct]

$\Rightarrow \mu_k v_k = 0$ (by ★) $\Rightarrow \mu_k = 0$ too.

Thus $v_1, \dots, v_k$ are lin ind, completing the induction. $\square$

Cor If $f : V \to V$ has $\dim V$ distinct eigenvalues, then f is diagonalizable.

With repeated eigenvalues, f might be diagonalizable, might not.

Note Now certain that diagonalization algorithm works!

Multiplicity

Defn A polynomial $p(x) \in F[x]$ splits over F when $\exists\, c, \lambda_1, \dots, \lambda_n \in F$ such that $p(x) = c(x-\lambda_1)\cdots(x-\lambda_n)$.

E.g. $x^2+1 = (x+i)(x-i)$ splits over $\mathbb{C}$ but not over $\mathbb{R}$.

Fundamental Theorem of Algebra Every $p(x) \in \mathbb{C}[x]$ splits over $\mathbb{C}$.

Prop Suppose $f: V \to V$ linear, $\dim V = n < \infty$. If f is diagonalizable, then $\chi_f(x)$ splits over F.

Pf If f is diagonalizable, then $\exists$ ordered basis α of V such that $A^\alpha_\alpha(f) = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_i \in F$.

Thus
$$\chi_f(x) = \det(D - xI_n) = (\lambda_1 - x)\cdots(\lambda_n - x)$$
$$= (-1)^n (x-\lambda_1)\cdots(x-\lambda_n). \quad \square$$

The converse fails: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $(x-1)^2$ but $\dim E_1 = 1$.

Defn Let $\dim V < \infty$. The geometric multiplicity of an eigenvalue λ of a linear transformation $f: V \to V$ is

$$\dim E_\lambda(f).$$

The algebraic multiplicity of λ is the number of times $(x-\lambda)$ divides $\chi_f(x)$.

Prop The geometric multiplicity of an eigenvalue of f is $\leq$ its algebraic multiplicity.

Pf Let $v_1, \dots, v_k$ be a basis of $E_\lambda(f)$, extend to a basis $v_1, \dots, v_n$ of V. The matrix for f wrt this basis looks like

$$A = \left(\begin{array}{c|c} \lambda I_k & B \\ \hline 0 & C \end{array}\right).$$

$f(v_i) = \lambda v_i$ for $1 \leq i \leq k$

Thus $\chi_f(x) = \det\left(\begin{array}{c|c} (\lambda-x)I_k & B \\ \hline 0 & C - xI_{n-k}\end{array}\right)$

$= (\lambda-x)^k \chi_C(x)$

induction + Laplace expansion:

$$\left(\begin{array}{c|c} \begin{matrix}\lambda-x & & \\ & \ddots & \\ & & \lambda-x\end{matrix} & B \\ \hline 0 & C-xI\end{array}\right)$$

so $k = \dim E_\lambda(f) \le$ algebraic multiplicity of λ.

$1 \le$ geom mult $\le$ alg mult

need $=$ for diagonalizability

Jordan normal form

A Jordan block of size k for $\lambda \in F$ is the $k \times k$ matrix

$$J_k(\lambda) := \begin{pmatrix}\lambda & 1 & & 0\\ & \lambda & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{pmatrix}.$$

E.g. $J_4(3) = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3\end{pmatrix}$, $J_1(\lambda) = (\lambda)$.

A matrix is in Jordan form when it is a "block sum" of Jordan blocks:

$$J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_r}(\lambda_r) := \begin{pmatrix} J_{k_1}(\lambda_1) & & & 0 \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{k_r}(\lambda_r) \end{pmatrix}$$

E.g.

$$J_2(2) \oplus J_2(2) \oplus J_1(5) \oplus J_3(4) = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & \\ & & \boxed{5} & \\ 0 & & & \boxed{\begin{matrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{matrix}} \end{pmatrix}.$$

Thm Suppose $\dim V < \infty$, $f: V \to V$ linear with $\chi_f(x)$ split over F. Then $\exists$ ordered basis α of V such that $A_\alpha^\alpha(f)$ is in Jordan form; the Jordan form of f is unique up to permutation of Jordan blocks.

Pf Use the structure theorem for finitely generated modules over a principal ideal domain — Math 332. $\square$

Note f is diagonalizable iff all its Jordan blocks have size 1.

$$U_n(\cos x) = \cos(nx)$$

Chebyshev polynomial of 1st type

$$U_n(x) = \det \begin{pmatrix} x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & & \\ & & & \ddots & \\ & & & & 2x \end{pmatrix}$$