28. X. 30

Goals

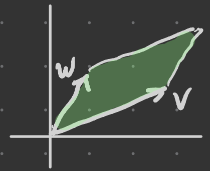
- signed volume via det
- signed volume and linear transformations.

Given vectors $v_1, \dots, v_n \in \mathbb{R}^n$, the **parallelepiped** spanned by $v_1, \dots, v_n$ is

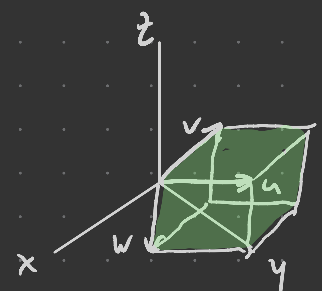
$$P(v_1, \dots, v_n) = \{ t_1 v_1 + \dots + t_n v_n \mid 0 \le t_i \le 1 \}.$$

E.g.

- In $\mathbb{R}^2$, $P(v, w) =$



- In $\mathbb{R}^3$, $P(u, v, w) =$

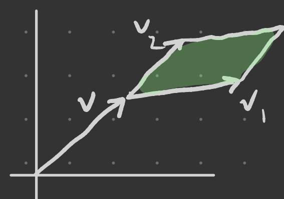


- In $\mathbb{R}^n$, $B_n := P(e_1, e_2, \dots, e_n) = [0,1]^n$ is the unit box.

Defn A parallelepiped in $\mathbb{R}^n$ is any set of the form

$$P(v_1, \dots, v_n) + v = \{ t_1 v_1 + \dots + t_n v_n + v \mid 0 \le t_i \le 1 \}.$$

E.g.



Note If $A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{n \times n}$, then $A \cdot B_n = P(v_1, \dots, v_n)$.

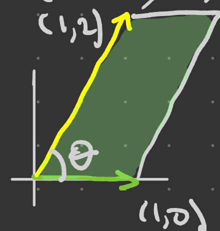
Indeed, $A \cdot (t_1 e_1 + t_2 e_2 + \dots + t_n e_n) = t_1 v_1 + \dots + t_n v_n$.

Signed area

$$SA : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(v,w) \longmapsto \begin{cases} \text{area}(P(v,w)) & \text{if ccw angle } v \text{ to } w \text{ is } \le \pi \\ -\text{area}(P(v,w)) & \text{if ccw angle } v \text{ to } w > \pi \end{cases}$$

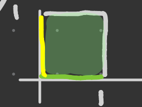
E.g. $SA((1,0),(1,2)) = \text{base} \times \text{height} = 2$

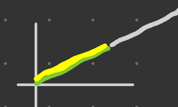


$\theta \le \pi$

Claim SA is multilinear, alternating, normalized and thus

$$SA(v,w) = \det\begin{pmatrix} - & v & - \\ - & w & - \end{pmatrix} = \det\begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix}.$$

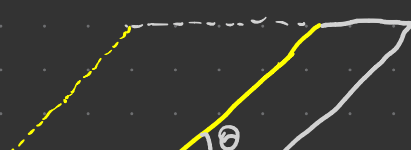
Pf • $SA(e_1, e_2) = \text{area}\Big($  $\Big) = 1$ so SA is normalized

• $SA(v, v) = \text{area}\Big($  $\Big) = 0$ so SA is alternating

• For $\lambda > 0$, $SA(\lambda v, w) = \lambda\, SA(v, w)$:

same angle
base scaled by $\lambda > 0$

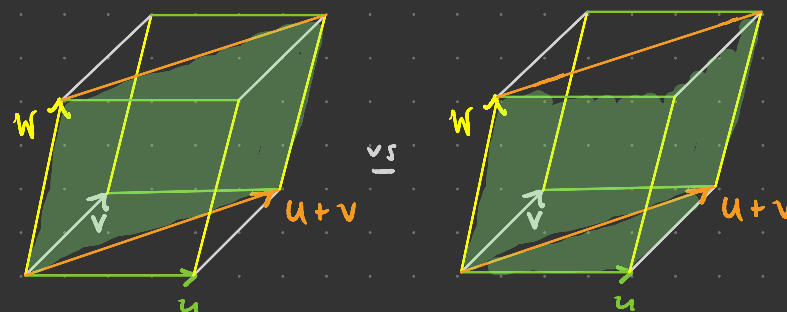
For $\lambda < 0$



angle from θ to $\pi + \theta$
base scaled by $|\lambda|$
so $SA(\lambda v, w) = \lambda SA(v, w)$.

Similar arguments give $SA(v, \lambda w) = \lambda SA(v,w)$.

- $SA(u+v, w) \overset{?}{=} SA(u,w) + SA(v,w)$:



implies $SA(u+v,w) = SA(u,w) + SA(v,w)$

(and similarly in 2nd variable). $\square$

<u>Defn</u> The volume of $P(v_1, \ldots, v_n)$ is

$$\left| \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \right|.$$

Thm For any parallelipiped $Q = P(v_1, \dots, v_n)$ and linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induced by a matrix A, $L(Q) = P(Av_1, \dots, Av_n)$ and $\mathrm{vol}(LQ) = |\det A|\,\mathrm{vol}\,Q$.

Pf Let $B = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$. Then $Q = B \cdot B_n$ and

$$LQ = A \cdot (B \cdot B_n) = (AB) \cdot B_n \quad \text{with } AB = \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix}$$

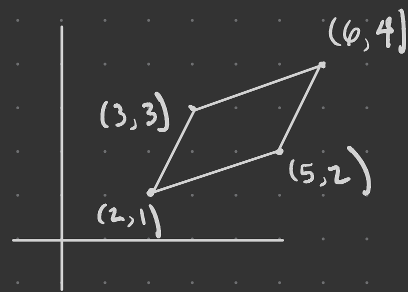
so $LQ = P(Av_1, \dots, Av_n)$. Now

$$\mathrm{vol}(LQ) = |\det(AB)| \overset{\text{det multiplicative}}{=} |\det(A) \cdot \det(B)|$$

$$\underset{|\,|\text{ is multiplicative}}{=} |\det A|\,\underbrace{|\det B|}_{\mathrm{vol}(Q)} = |\det A|\,\mathrm{vol}(Q). \quad \square$$

Rmk We proved that L transforms volumes of parallelepipeds by $|\det A|$ (for $L = \mathrm{map}_A$).
In Math 202, you will generalize this to other volumes.

Problem Find the area of this figure:



Answer This is $P((3,1),(1,2)) + (2,1)$ so its area is

$$\left|\det\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\right| = |6-1| = 5.$$

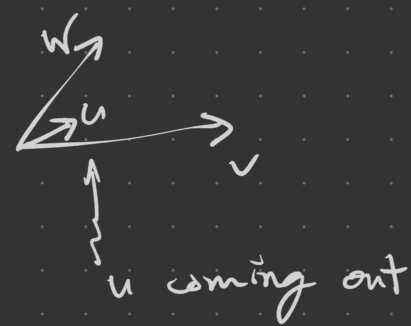
Problem What is the volume of λB_n, $\lambda > 0$?

Answer $$\mathrm{vol}(\lambda \cdot B_n) = \left|\det\begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots & \\ & & & \lambda \end{pmatrix}\right| = |\lambda^n| = \lambda^n$$

Orientation Call an ordered basis $(v_1, \dots, v_n)$ of $\mathbb{R}^n$ positively oriented when $\det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} > 0$;

if $\det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} < 0$ call it negatively or'd.

Note $\mathcal{E}_n = (e_1, \dots, e_n)$ is positively or'd



(v, w, u) is pos or'd in $\mathbb{R}^3$

In 2D, angle condition.