24.X.28

**Goals**
- Laplace expansion of det
- Existence & uniqueness of det

**Defn** For $A \in F^{n\times n}$, $1 \le i, j \le n$, set $A^{ij} \in F^{(n-1)\times(n-1)}$ to be the matrix created by deleting the $i$-th row and $j$-th column from $A$.

**E.g.** If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$, then $A^{23} = \begin{pmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{pmatrix}$

not 23rd power

**Thm** (Laplace expansion) Suppose $A \in F^{n\times n}$ and $1 \le k \le n$. Then

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} A_{kj} \det(A^{kj})$$

$(k,j)$ minor

with sign $(k,j)$ cofactor

expansion of det $A$ along the $k$-th row of $A$

E.g.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (-1)^{1+1} a \det(d) + (-1)^{1+2} b \det(c)$$

$$= ad - bc \quad \checkmark$$

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 6 & 7 \end{pmatrix} = (-1)^{2+1} \cdot 0 \det\begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix} + (-1)^{2+2} \cdot 4 \det\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$$

$$+ (-1)^{2+3} \cdot 0 \cdot \det\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

$$= 4\,(1\cdot 7 - 3\cdot 5) = -32.$$

<u>Note</u> $(-1)^{k+j}$ makes a "sign checkerboard":

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This can be a helpful mnemonic when expanding along some row.

<u>Cor</u> [Laplace expansion along columns] For $A \in F^{n\times n}$, $1 \le k \le n$,

$$\det A = \sum_{i=1}^{n} (-1)^{i+k} A_{ik} \det(A^{ik})$$

<u>Pf</u> We know $\det A = \det A^T$ and this is Laplace exp'n along the $k$-th row of $A^T$.

**Question** What is the determinant of $\begin{array}{cc} \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 6 \\ 4 & 7 & 0 \end{pmatrix} & \begin{array}{c} + \\ - \\ + \end{array} \end{array}$ ?

$$-6 \cdot \det \begin{pmatrix} 1 & 3 \\ 4 & 7 \end{pmatrix} = -6 \cdot (-5) = 30$$

**Pf sketch for Laplace Expansion Thm** By permutation exp'n,

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}.$$

Factor out all the $A_{kj}$ terms:

$$\det A = \sum_{j=1}^{n} A_{kj} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(k)=j}} \operatorname{sgn}(\sigma) \prod_{\substack{1 \le i \le n \\ i \ne k}} A_{i\sigma(i)}$$

Need to show this is

$(-1)^{k+j} \det A^{kj}$

The part has all the correct $A_{ij}$ terms. Must check that the sign is off by $(-1)^{k+j}$ (hint: swap rows). $\square$

**Thm** [$\exists!$ det] There is a unique multilinear, alternating, normalized function $\det : F^{n\times n} \longrightarrow F$.

**Pf** Inspired by Laplace expansion, make the following recursive definition of a function $d: F^{n\times n} \longrightarrow F$:

$n=1$ $\quad d(a)=a$

$n>1$ $\quad d(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j}\, d(A^{\hat{1}\hat{j}})$.

Check $d$ is multilin, alt, normalized.

Then $d$ is a well-defined determinant function.

Our work with det & row operations shows that all determinant functions are determined by a sequence of row operations to rref:

if $E_1, \ldots, E_\ell$ elementary s.t. $\operatorname{rref}(A) = E_\ell \cdots E_1 A$

then $\det A = \dfrac{\det(\operatorname{rref}(A))}{\det(E_\ell)\cdots\det(E_1)}$.

Thus any two det functions are in fact the same! $\square$

Time permitting, the general linear group:

Let $F^\times := F \setminus \{0\}$, $GL_n(F) \subseteq F^{n\times n}$ be the invertible $n\times n$ matrices.

Then $GL_n(F) = \det^{-1} F^\times = \{A \in F^{n\times n} \mid \det A \in F^\times\}$

$$\Updownarrow$$

$\det A \neq 0$

$$\begin{tikzcd} F^{n\times n} \arrow[r, "\det"] & F \\ GL_n(F) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\det"'] & F^\times \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

(group under matrix multiplication

## #3

$$\mathbb{R}^2 \xrightarrow{t} \mathbb{R}^2$$

$$e_1 \longmapsto (1,1) + 3(1,-2) = (4,-5)$$

$$e_2 \longmapsto 2(1,1) + 4(1,-2) = (6,-6)$$

$$A^{\beta}_{\mathcal{E}}(t) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\beta = ((1,1), (1,-2))$$

$$\Longrightarrow \quad A^{\mathcal{E}}_{\mathcal{E}}(t) = \begin{pmatrix} 4 & 6 \\ -5 & -6 \end{pmatrix}$$

$$\delta = ((0,1), (1,1)) \quad \gamma = ((-1,0), (2,1))$$

$$\varepsilon = ((1,2), (1,0)) \quad \zeta = ((1,2), (2,1))$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "t"] \arrow[d, shift right, "\mathrm{Rep}_\delta"'] & \mathbb{R}^2 \arrow[d, shift left, "\mathrm{Rep}_\gamma"] \\
\mathbb{R}^2 \arrow[r, "A^{\gamma}_{\delta}(t)"'] \arrow[u, shift right, "P"'] & \mathbb{R}^2 \arrow[u, shift left, "Q"]
\end{tikzcd}$$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A_\gamma^\gamma(t) = Q^{-1} A_\xi^\xi(t) P$$

$$= \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ -5 & -6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 10 \\ -6 & -11 \end{pmatrix}$$

$$= \begin{pmatrix} -18 & -32 \\ -6 & -11 \end{pmatrix}$$

$$t(0,1) = (6,-6)$$

$$t(1,1) = t(e_1) + t(e_2)$$

$$= (4,-5) + (6,-6)$$

$$= (10,-11)$$

$$-18 \cdot (-1,0) - 6(2,1) = (6,-6) \checkmark$$

$$-32 \cdot (-1,0) - 11(2,1) = (10,-11) \checkmark$$